

SCHLOSS DAGSTUHL

INTERNATIONALES
BEGEGNUNGS-
UND FORSCHUNGSZENTRUM
FÜR INFORMATIK

Kevin Compton, Jean-Eric Pin ,
Wolfgang Thomas (editors):

Automata Theory: Infinite Computations

Dagstuhl-Seminar-Report; 28
6.-10.1.92 (9202)

SEMINAR-REPORT

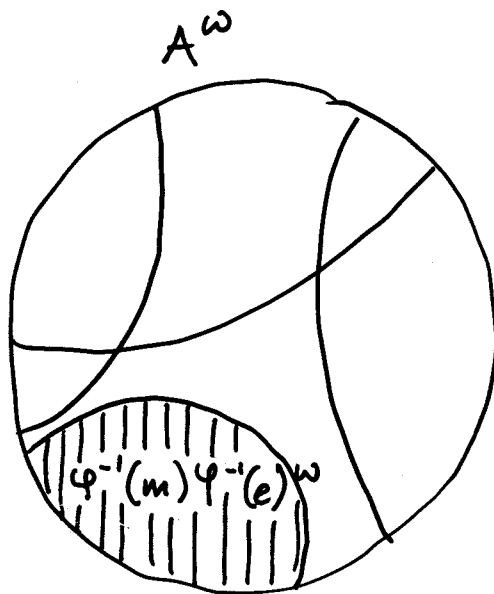
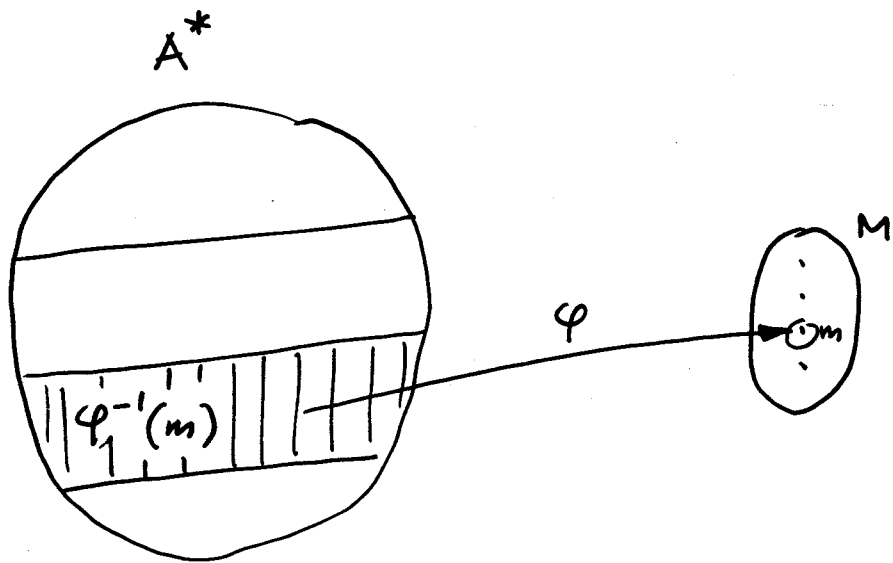
Adding an Infinite Product to a Semigroup

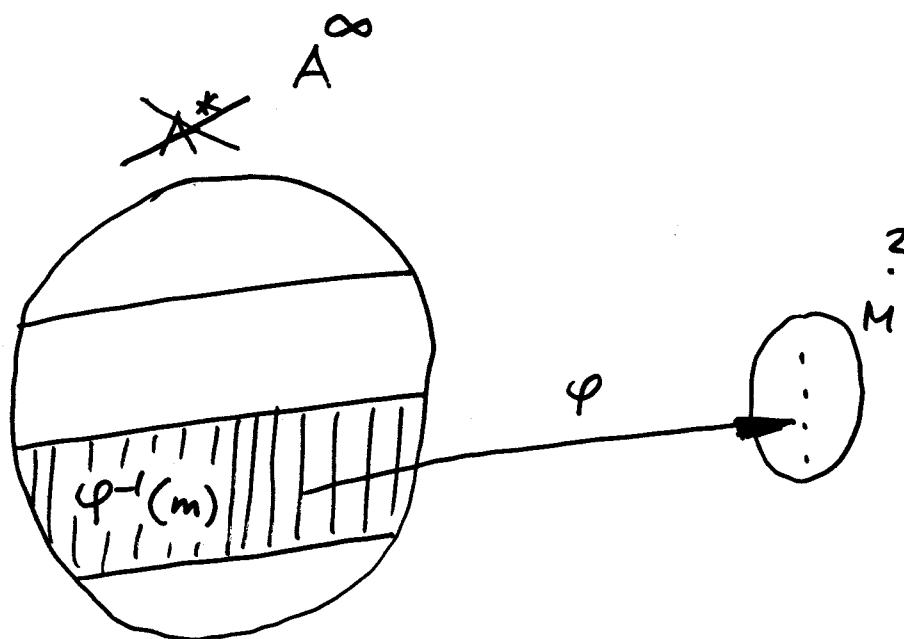
R.R. Redziejowski (Lidingö)

A semigroup (S, \cdot) is extended with an ω -argument operation $\Pi: S^{\mathbb{N}} \rightarrow Q$ where Q is a certain set, possibly disjoint with S . The operation Π is required to satisfy three axioms, expressing generalized associativity compatible with the semigroup operation.

It follows from the axioms and the Ramsey lemma that for a finite S , Π can have only a finite number of distinct values, all reducible to the form se^{ω} where $s, e \in S$ and $e^2 = e$.

Homomorphism and a unique free algebra are defined for the class of semigroups extended with the operation Π . The purpose of the construction is to express recognizability of ω -languages in a way analogous to that of finite-word languages (directly in terms of a homomorphism into a finite algebraic structure).





Problem: How to define φ ?

Words in A^ω are infinite products of letters.

These products must be mirrored in M .

What is the product of a sequence of elements of M ?

Ex. $M = (\{a, b\}, \cdot)$

What is $b \cdot a \cdot a \cdot a \cdot a \cdot \dots$?

a	b
a	b
b	b

a (has almost only a)

b (any initial finite product gives b)

Given semigroup (S, \cdot)

Add ω -product (Q, π)

where:

- Q is any set

- $\pi: \text{Seq}(S) \rightarrow Q$ a function onto

Example 1

$S = (A^+, \cdot)$ for some alphabet A

$$Q = A^\omega$$

$\Pi(w_1, w_2, w_3, \dots) = w_1 w_2 w_3 \dots$ (concatenation)

$(w_i \in A^+)$.

Example 2

$S = (A^*, \cdot)$

$Q = A^\omega \cup A^* z$ where $z \notin A$

If $w_1, w_2, w_3, \dots = w_1, w_2, \dots, w_n, 1, 1, 1, \dots$

for some n , then

$$\Pi(w_1, w_2, w_3, \dots) = w_1 w_2 \dots w_n z$$

Otherwise

$$\Pi(w_1, w_2, w_3, \dots) = w_1 w_2 w_3 \dots$$

Example 3

Same as Example 2, but

$$\Pi(w_1, w_2, \dots, w_n, 1, 1, 1, \dots) = w_1 w_2 \dots w_n$$

$$(Q = A^\omega \cup A^*)$$

Example 4

$S = (2^X, \cup)$ for some space X

$$Q = 2^X$$

$$\pi(x_1, x_2, x_3, \dots) = \bigcup_{i \in \mathbb{N}} x_i$$

(where $x_i \subseteq X$).

Example 5

$S = (\{a, b\}, \cdot)$ where \cdot is

a	b
a	ab
b	bb

$$Q = \{a, b\}$$

For $x \in \text{Seq}(S)$:

$$\pi(x) = \begin{cases} a & \text{if } x = a, a, a, \dots \\ b & \text{otherwise} \end{cases}$$

Example 6

$S = (\{0, 1\}, \cup)$

$$Q = \{A, B\}$$

For $x \in \text{Seq}(S)$:

$$\pi(x) = \begin{cases} A & \text{if } x = 0, 0, 0, \dots \\ B & \text{otherwise} \end{cases}$$

Example 7

$$S = (\mathbb{R}_+, +) \quad \mathbb{R}_+ - \text{real numbers } \geq 0$$

$$Q = \mathbb{R}_+ \cup \{\infty\}$$

For $r = r_1, r_2, r_3, \dots$ ($r_i \in \mathbb{R}_+$):

$$\pi(r) = \begin{cases} \sum r_i & \text{if } r \text{ converges} \\ \infty & \text{otherwise} \end{cases}$$

Example 8

Same as Ex 7, but \mathbb{R} instead of \mathbb{R}_+

Example 9

$$S = (\{0, 1\}, \vee)$$

$$Q = \{0, 1\}$$

For $x \in \text{Seq}(S)$:

$$\pi(x) = \begin{cases} 1 & \text{if } x = 0, 0, 0, \dots \\ 0 & \text{otherwise} \end{cases}$$

Example 10

$$S = (\{0, 1\}, \vee)$$

$$Q = \{A, B\}$$

For $x \in \text{Seq}(S)$:

$$\pi(x) = \begin{cases} A & \text{if } x \text{ contains none or inf. many 1's} \\ B & \text{otherwise} \end{cases}$$

- What is wrong with Example 10?

$$\begin{aligned}\bar{\pi}(0, 0, 0, \dots) &= A = \pi(1, 1, 1, \dots) \\ \pi(1, 0, 0, 0, \dots) &= \underline{B} \neq \pi(1, 1, 1, 1, \dots)\end{aligned}$$

→ We want $\pi(x) = \bar{\pi}(y) \Rightarrow \pi(s, x) = \bar{\pi}(s, y)$

- What is wrong with Example 9?

$$\begin{aligned}\pi(1, 0, 0, 0, \dots) &= 0 \\ \swarrow \\ 1 \vee \pi(0, 0, 0, \dots) &= 1 \vee 1 = 1 \neq 0\end{aligned}$$

→ We want $\pi(x) \in S \Rightarrow s \cdot \pi(x) = \pi(s, x)$

- What is wrong with Example 8?

$$\begin{aligned}\pi(-1, +1, -1, +1, \dots) &= \infty \\ \pi((-1+1), (-1+1), \dots) &= \pi(0, 0, 0, \dots) = 0 \neq \infty\end{aligned}$$

→ We want "infinite associativity"

Definition

Let $x \in \text{Seq}(S)$

$n \in \text{Seq}(\mathbb{N})$ be strictly increasing

$x|n$ (x reduced by n) is a sequence $y \in \text{Seq}(S)$

defined by

$$y_i = \begin{cases} x_1 \cdot x_2 \cdot \dots \cdot x_{n_1} & \text{for } i=1 \\ x_{n_{i-1}+1} \cdot x_{n_{i-1}+2} \cdot \dots \cdot x_{n_i} & \text{for } i > 1 \end{cases}$$

Examples

$$S = \{a, b\} \quad \begin{array}{c|c} a & b \\ a & ab \\ b & bb \end{array}$$

$$(a, b, a, b, a, b, \dots) | (3, 6, 9, \dots) = a \cdot b \cdot a, b \cdot a \cdot b, \dots \\ = b, b, b, \dots$$

$$(a, b, a, b, a, b, \dots) \times (2, 3, 5, 6, \dots) = a \cdot b, a, b \cdot a, b, \dots \\ = b, a, b, b, \dots$$

$x|n$ is a reduction of x

x is an expansion of $x|n$

$$(A1) \quad \pi(x) = \pi(y) \Rightarrow \pi(s, x) = \pi(s, y)$$

FOR $x, y \in \text{Seq}(S), s \in S$

$$(A2) \quad \pi(x) \in S \Rightarrow s \cdot \pi(x) = \pi(s, x)$$

FOR $x \in \text{Seq}(S), s \in S$

(A3) π IS INFINITELY ASSOCIATIVE

$$(A3) \quad \pi(x) = \pi(x|n)$$

FOR $x \in \text{Seq}(X)$

$n \in \text{Seq}(N)$

n STRICTLY ASCENDING

Definition:

Extended semigroup (S, Q, \cdot, π)

- (S, \cdot) is a semigroup

- (Q, π) is an ω -product over (S, \cdot) satisfying (A1)-(A3)

If (A1) holds,
for each $s \in S$, $q \in Q$ exists unique
 $s \circ q = \pi(s, x)$ where $\pi(x) = q$.

Write informally $\pi(x_1, x_2, x_3 \dots)$ as $x_1 \circ x_2 \circ x_3 \circ \dots$

(A1) permits things like this:

$$x_1 \circ x_2 \circ x_3 \circ \dots = x_1 \circ (x_2 \circ (x_3 \circ x_4 \circ \dots))$$

(A2) permits things like this:

$$x_1 \circ x_2 \circ x_3 \circ \dots = (x_1 \circ x_2) \circ (x_3 \circ x_4 \circ \dots)$$

(A3) permits things like this:

$$x_1 \circ x_2 \circ x_3 \circ \dots = (x_1 \circ x_2) \circ (x_3 \circ x_4) \circ \dots$$

Definition

Let $x, y \in \text{Seq}(S)$.

$x \sim y$ (x and y are similar)

means: x can be transformed into y by a sequence of reductions and/or expansions.

Properties

- \sim is an equivalence
- $x \sim y \Rightarrow s, x \sim s, y$ for any $s \in S$
- If $\varphi: S \rightarrow S'$ is a homomorphism then $x \sim y \Rightarrow \varphi x \sim \varphi y$
- (From Ramsey)
If S is finite then for every $x \in \text{Seq}(S)$ exists reduction $x/n = s, e, e, e, \dots$ where e is an idempotent of S
- If S is finite then the number of equivalence classes of \sim is finite

Consequences

- (A3) is equivalent to $x \sim y \Rightarrow \pi(x) = \pi(y)$
- In any extended semigroup (S, Q, \cdot, π) , if S is finite, so is Q .

Definition

Let (S, Q, \cdot, π) , (S', Q', \circ, π')
be extended semigroups.

$$\varphi: (S \cup Q) \rightarrow (S' \cup Q')$$

is a homomorphism if:

$$- \varphi(S) \subseteq S'$$

$$- \varphi(Q) \subseteq Q'$$

$$- \varphi(s_1 \cdot s_2) = \varphi(s_1) \circ \varphi(s_2) \quad \text{for } s_1, s_2 \in S$$

$$- \varphi(\pi(x)) = \pi'(\varphi(x)) \quad \text{for } x \in \text{Seq}(S)$$

Definition

An ω -product (Q, π) over a semigroup (S, \cdot) is free if:

- $S \cap Q = \emptyset$
- $x \sim y \iff \pi(x) = \pi(y)$ for all $x, y \in \text{Seq}(S)$

Properties

- A free ω -product satisfies (A1-A3)
- A free ω -product over a given (S, \cdot) is unique (up to isomorphism)
- Let (S, Q, \cdot, π) be an ^{extended} semigroup such that (Q, π) is free.
Let (S', Q', \cdot, π') be any extended semigroup.
Then, each homomorphism $\varphi: S \rightarrow S'$ has a unique extension to homomorphism $(S \cup Q) \rightarrow (S' \cup Q')$.

Should be easy to prove
something like:

$L \subseteq A^\infty$ is recognizable

iff

there exists finite extended semigroup

(S, Q, \cdot, π) and a homomorphism

$\varphi: A^\infty \rightarrow S \cup Q$

such that L is a union of classes

$\varphi^{-1}(s)$ for some $s \in S \cup Q$.