# On Associative Omega-Products*

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**Abstract**

In recent years, a number of classical results connecting rational languages with finite semigroups have been extended to infinite-word languages using the notion of an *ω-semigroup*: a semigroup augmented with an associative infinite product. This paper takes a closer look at the associative infinite product itself. It suggests some improvements and presents a couple of new facts.

## 1 Introduction

Extending binary operation to an infinite sequence of operands is not a new idea. A classical example is the infinite series, which is such an extension of "+". A newer example is infinite concatenation of words, a daily bread in the study of automata on infinite words. Another is concatenation product of an infinite sequence of sets of words. Some of these extensions are associative, that is, the result does not change if the factors are grouped by parentheses. Some are not, like the infinite series in the domain of all real numbers. For a long time, associativity has been exploited in a rather informal way. But in recent years, the research connecting automata, semigroups, and infinite-word languages required a more formal treatment of infinite associativity. The products appearing in that context do not have the intuitive form of being the limit of longer and longer finite products; thus the need of a precise treatment.

It seems that the first formal definition of infinite associativity was published in [10]. Slightly before, the present author proposed a set of axioms for an associative infinite product in a Dagstuhl Seminar lecture, of which only an abstract [17] was published.

The associative infinite product was introduced in [10] as a component of an *ω-semigroup*: a semigroup augmented with such a product. Using this new notion, one could extend to infinite-word languages a number of classical results connecting rational languages with finite semigroups [4–6, 10–14].

This paper takes a closer look at the associative infinite product itself. We suggest some improvements and present a couple of new facts.

In [10] and all subsequent work using ω-semigroups, one aspect of associativity is ensured by postulating existence of an additional associative operation, the "mixed product". We note that this postulate can be replaced by an equivalent property of the infinite product itself, which may sometimes be more convenient. In all work with ω-semigroups, values of the infinite product are distinct from elements of the underlying semigroup. This is not true in the general case, and we consider that possibility. A number of important facts about the case of finite semigroup have been so far stated and proved as properties of homomorphisms. We note that they can be stated in terms of the infinite product itself.

We consider different associative infinite products that can be defined on the same semigroup, and introduce the notion of a homomorphism between these products. It turns out that all such products are homomorphic images of a "free product", and one-to-one homomorphic images of certain "primary products".

It was customary in the past to avoid concatenation of an infinite sequence of empty words. We give examples showing that it may be usefully incorporated in an associative infinite product, and in more than one way.

We also take a short look at infinite products defined as a limit of partial products, and at infinite products in presence of left zeros.

The infinite products considered here are products of sequences indexed by natural numbers. Some recent papers [1, 2] introduce associative products of sequences indexed by higher ordinals. A general theory of infinitary operations was developed as early as 1959 by Słomiński [19], but with a focus on properties other than associativity.

*Appeared in *Fundamenta Informaticae 60* (2004) 333–350.

## 2 Preliminaries

We use terminology and notation from [7] and [23], adapted to our purpose. The image of $x \in X$ under mapping $\varphi : X \to Y$ is denoted by $\varphi(x)$. The composition of mappings $\varphi$ and $\psi$ is denoted by $\varphi\psi$, with $\varphi\psi(x)$ meaning $\psi(\varphi(x))$.

An *operation* on set $X$ is a mapping $\cdot : X \times X \to X$. The image of $(x, y) \in X \times X$ under such mapping is denoted by $x \cdot y$. The operation is *associative* if $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in X$.

An *action* of set $S$ on set $X$ is a mapping $\circ : S \times X \to X$. The image of $(s, x) \in S \times X$ under such mapping is denoted by $s \circ x$.

We say that mapping $\varphi : X \to Y$ is *compatible* with action $\circ$ to mean that $\varphi(x_1) = \varphi(x_2)$ implies $\varphi(s \circ x_1) = \varphi(s \circ x_2)$ for all $s \in S$ and $x_1, x_2 \in X$. The *image* of $\circ$ under $\varphi$ is an action $\circ_\varphi$ of $S$ on $Y$ such that $s \circ_\varphi \varphi(x) = \varphi(s \circ x)$ for all $s \in S$ and $x \in X$. If such an image exists, it is clearly unique. The following facts are easy to verify:

**Lemma 1.** *For any action $\circ$ on $X$, surjective mapping $\varphi : X \to Y$, and mapping $\psi : Y \to Z$:*
*(a) An image of $\circ$ under $\varphi$ exists if and only if $\varphi$ is compatible with $\circ$.*
*(b) If $\varphi$ is compatible with $\circ$, $\varphi\psi$ is compatible with $\circ$ if and only if $\psi$ is compatible with $\circ_\varphi$.*
*(c) The image $\circ_{\varphi\psi}$, if it exists, is identical to $(\circ_\varphi)_\psi$.*

For an equivalence relation $\approx$ on set $X$, its *natural mapping* $\mathsf{nat}_\approx$ assigns to each $x \in X$ the class of $X/\approx$ containing $x$. We say that the relation $\approx$ is compatible with action $\circ$ of $S$ on $X$ if its natural mapping is compatible with $\circ$; in other words, if $x \approx y \Rightarrow s \circ x \approx s \circ y$ for all $s \in S$ and $x, y \in X$.

The set of all natural numbers (positive integers) is denoted by $\mathbb{N}$. A sequence $\mathbf{x}$ of elements of set $X$ is a mapping $\mathbf{x} : \mathbb{N} \to X$. It is visualized as a linear arrangement of elements $\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \dots$ . The set of all sequences of elements of $X$ is denoted by $X^\mathbb{N}$. The sequence $x, x, x, \dots$ with all elements equal to $x \in X$ is denoted by $x^\mathbb{N}$. The sequence $x, \mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \dots$ obtained by adding $x \in X$ in front of sequence $\mathbf{x} \in X^\mathbb{N}$ is denoted by $x \circ \mathbf{x}$. This action $\circ : X \times X^\mathbb{N} \to X^\mathbb{N}$ is referred to as *prefixing*.

A *semigroup* is a pair $(S, \cdot)$ where $S$ is a set and $\cdot$ is an associative operation on $S$. We refer to that operation as the *semigroup product*. In the following discussion, $\mathbf{S}$ denotes an arbitrary semigroup $(S, \cdot)$.

## 3 Infinite associativity

We define an *infinite product*, or *$\omega$-product*, on $\mathbf{S}$ as a pair $(V, \pi)$ where $V$ is a set, not necesarily disjoint with $S$, and $\pi$ is a surjective mapping from $S^\mathbb{N}$ to $V$. We often speak informally of mapping $\pi$ as *the* infinite product, and refer to $V$ as the set of values of the product. This definition has yet no relation to the semigroup product of $\mathbf{S}$. In particular, $\pi(s_1, s_2, s_3, \dots)$ is not required to be the limit of $s_1 \cdot s_2 \cdot \ldots \cdot s_n$ for $n \to \infty$. In the following, we connect the two products by extending the notion of associativity.

The important consequence of associativity of semigroup product is that products of more than two factors can be written unambiguously as $s_1 \cdot s_2 \cdot \ldots \cdot s_n$. The result remains unchanged if factors are grouped in an arbitrary way by means of parentheses.

It is convenient to write the $\omega$-product $\pi(s_1, s_2, s_3, \dots)$ symbolically as $s_1 \cdot s_2 \cdot s_3 \cdot \ldots$ . In this form, the symbol $\cdot$ denotes the $\omega$-product, not an operation on two neighbouring factors. In an analogy to finite products, we would like the value of infinite product to remain unchanged by insertion of parentheses. We would like to freely use identities such as

$$s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \cdots = (s_1 \cdot s_2 \cdot s_3) \cdot (s_4 \cdot s_5 \cdot s_6) \cdot \ldots, \tag{1}$$

$$s_1 \cdot s_2 \cdot s_3 \cdot \ldots = (s_1 \cdot \ldots \cdot s_n) \cdot (s_{n+1} \cdot s_{n+2} \cdot s_{n+3} \cdot \ldots). \tag{2}$$

In (1), we understand the symbol $\cdot$ within the parentheses to mean the semigroup product, and outside the parentheses to mean the $\omega$-product. In (2), the symbol $\cdot$ within the first pair of parentheses denotes the semigroup product; within the second pair, it denotes the $\omega$-product. The dot in the middle is neither of the two: it stands for an action $S \times V \to V$. This action will have to be suitably defined.

### 3.1 Invariance under contractions

We need an additional definition to formalize (1). Let $\mathbf{x} \in S^{\mathbb{N}}$ and let $\mathbf{n} = n_1, n_2, n_3, \dots$ be an ascending sequence of natural numbers. Define the *contraction* of $\mathbf{x}$ by $\mathbf{n}$, written $\mathbf{x}|\mathbf{n}$, to be the sequence $\mathbf{y} \in S^{\mathbb{N}}$ where:

$$\mathbf{y}(1) = \mathbf{x}(1) \cdot \ldots \cdot \mathbf{x}(n_1),$$
$$\mathbf{y}(i) = \mathbf{x}(n_{i-1}+1) \cdot \ldots \cdot \mathbf{x}(n_i) \quad \text{for } i > 1.$$

For example: $(s_1, s_2, s_3, \dots) \mid (1, 3, 5, \dots) = (s_1), (s_2 \cdot s_3), (s_4 \cdot s_5), \dots$. (To simplify notation, we allow $k = 1$ in a product of $k$ factors; the product is then identical to the single factor.) For sequences $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$, we write $\mathbf{x} \rhd \mathbf{y}$ or $\mathbf{y} \lhd \mathbf{x}$ to mean that $\mathbf{y}$ is a contraction of $\mathbf{x}$ (by some $\mathbf{n}$). The desired property can now be expressed as:

$$\mathbf{x} \rhd \mathbf{y} \Rightarrow \pi(\mathbf{x}) = \pi(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}. \tag{3}$$

### 3.2 Compatibility with prefixing

To have a valid identity (2), we must define an action $\bullet : S \times V \to V$ such that

$$\pi(s_1, s_2, s_3, \dots) = (s_1 \cdot \ldots \cdot s_n) \bullet \pi(s_{n+1}, s_{n+2}, s_{n+3}, \dots) \tag{4}$$

for each sequence $s_1, s_2, s_3, \dots \in S^{\mathbb{N}}$ and $n \geq 1$. The action must have this property, as a special case for $n = 1$:

$$s \bullet \pi(\mathbf{x}) = \pi(s \circ \mathbf{x}) \quad \text{for all } s \in S \text{ and } \mathbf{x} \in S^{\mathbb{N}}. \tag{5}$$

That means $\bullet$ must be the image of $\circ$ under $\pi$. According to Lemma 1(a), such an image exists if and only if $\pi$ is compatible with $\circ$, that is,

$$\pi(\mathbf{x}) = \pi(\mathbf{y}) \Rightarrow \pi(s \circ \mathbf{x}) = \pi(s \circ \mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in S^{\mathbb{N}} \text{ and } s \in S. \tag{6}$$

The action $\bullet$ defined by (5), if it exists, is in the following called the *mixed product* induced by $\pi$. If (3) holds, the mixed product satisfies (4) also for $n > 1$ because $\pi(s_1, s_2, s_3, \dots)$ is then equal to $\pi((s_1 \cdot \ldots \cdot s_n),\, s_{n+1},\, s_{n+2},\, s_{n+3}, \dots)$.

### 3.3 Consistency with semigroup product

As just shown, an $\omega$-product invariant under contractions and compatible with prefixing satisfies the extended associativity rules (1) and (2) with a suitable interpretation of operators. However, this interpretation is ambiguous if the value of $\omega$-product $(s_{n+1} \cdot s_{n+2} \cdot s_{n+3} \cdot \dots)$ in (2) is an element of $S$. In that case, the middle operator could either be the semigroup product or the mixed product. To avoid the ambiguity, we want these two products to be identical, that is, we require:

$$s \bullet t = s \cdot t \quad \text{for all } s, t \in S. \tag{7}$$

By definition (5) of mixed product, this is equivalent to:

$$\pi(s \circ \mathbf{x}) = s \cdot \pi(\mathbf{x}) \quad \text{for all } s \in S \text{ and } \mathbf{x} \in S^{\mathbb{N}} \text{ such that } \pi(\mathbf{x}) \in S. \tag{8}$$

This condition implies that $S \cap V$ must be a left ideal of $S$.

### 3.4 Associative $\omega$-product

In the following, an $\omega$-product $(V, \pi)$ on a semigroup $(S, \cdot)$ is called *associative* if it has these three properties:

(A1) $\mathbf{x} \rhd \mathbf{y} \;\Rightarrow\; \pi(\mathbf{x}) = \pi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$.
(A2) $\pi(\mathbf{x}) = \pi(\mathbf{y}) \;\Rightarrow\; \pi(s \circ \mathbf{x}) = \pi(s \circ \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$ and $s \in S$.
(A3) $\pi(s \circ \mathbf{x}) = s \cdot \pi(\mathbf{x})$ for all $s \in S$ and $\mathbf{x} \in S^{\mathbb{N}}$ such that $\pi(\mathbf{x}) \in S$.

We recall that (A1) and (A2) were chosen to ensure validity of arbitary equations of the form (1) and (2); (A3) ensures that (2) is not ambiguous. Notice that (A3) implies (A2) whenever $\pi(\mathbf{x}) \in S$.

### 3.5 Relation to Perrin-Pin $\omega$-semigroup

Perrin and Pin [10–12] define an *$\omega$-semigroup* as a two-sorted algebra $(S, V, \cdot, \bullet, \pi)$ on disjoint sets $S$ and $V$, with three operations:

- finite product $\cdot : S \times S \to S$,
- mixed product $\bullet : S \times V \to V$,
- infinite product $\pi : S^{\mathbb{N}} \to V$.

The operations have these properties:

(PP1) $(S, \cdot\,)$ is a semigroup.
(PP2) $(s \cdot t) \bullet u = s \bullet (t \bullet u)$ for all $s, t \in S, u \in V$.
(PP3) $\mathbf{x} \rhd \mathbf{y} \;\Rightarrow\; \pi(\mathbf{x}) = \pi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$.
(PP4) $\pi(s \circ \mathbf{x}) = s \bullet \pi(\mathbf{x})$ for all $s \in S$ and $\mathbf{x} \in S^{\mathbb{N}}$.

The algebra is *complete* if the mapping $\pi$ is surjective.

One can easily see that a complete $\omega$-semigroup $\mathbf{S}_\omega = (S, V, \cdot, \bullet, \pi)$ can be alternatively defined as a pair consisting of a semigroup $(S, \cdot\,)$ and an associative $\omega$-product $(V, \pi)$ on that semigroup, with $S \cap V = \varnothing$. The infinite product of $\mathbf{S}_\omega$ is the mapping $\pi$ of $(V, \pi)$, and the mixed product is the mixed product induced by $\pi$. (PP2) follows from (A1), (A2), and $\pi$ being surjective; (PP3) is identical to (A1) and (PP4) follows from (A2).

One can also see that the pair $(V, \pi)$ of a complete $\omega$-semigroup $(S, V, \cdot, \bullet, \pi)$ is an associative $\omega$-product on $(S, \cdot\,)$.

### 3.6 Extension to subsets

In many applications, it is useful to extend the semigroup product to subsets by defining, for $X$, $Y \subseteq S$:

$$X \cdot Y = \{x \cdot y \,|\, x \in X \text{ and } y \in Y\}.$$

The extended operation is an associative operation on $2^S$, and defines a new semigroup $(2^S, \,\cdot\,)$. An associative $\omega$-product $(V, \pi)$ on $\mathbf{S}$ can be similarly extended to an $\omega$-product $(2^V, \pi)$ on $(2^S, \,\cdot\,)$ by defining, for a sequence $\mathbf{X} = X_1, X_2, X_3, \ldots$ of subsets of $S$:

$$\pi(\mathbf{X}) = \{\pi(x_1, x_2, x_3 \ldots) \,|\, x_i \in X_i \text{ for } i \geq 1\}.$$

Using standard techniques, one can show that if $(V, \pi)$ is associative, so is its extension $(2^V, \pi)$.

## 4 Mappings of $\omega$-products

Let $(V, \pi)$ and $(\bar{V}, \bar{\pi})$ be two associative $\omega$-products on $\mathbf{S}$. We say that mapping $\varphi : V \to \bar{V}$ is a *homomorphism* from $(V, \pi)$ to $(\bar{V}, \bar{\pi})$ if $\bar{\pi} = \pi\varphi$ and $\varphi(S \cap V) \subseteq S$. Because the mapping $\bar{\pi}$ in an $\omega$-product $(\bar{V}, \bar{\pi})$ is surjective, so must be $\varphi$. Therefore we speak of $(\bar{V}, \bar{\pi})$ as a *homomorphic image* of $(V, \pi)$.

We say that products $(V, \pi)$ and $(\bar{V}, \bar{\pi})$ are *isomorphic* if they are homomorphic images of each other. Using properties of surjective mappings, one can verify that the two homomorphisms must be then bijections and inverses of each other, and must map $S \cap V$ and $S \cap \bar{V}$ onto each other.

Written explicitly, $\bar{\pi} = \pi\varphi$ means that $\bar{\pi}(\mathbf{x}) = \varphi(\pi(\mathbf{x}))$ for all $\mathbf{x} \in S^{\mathbb{N}}$, which is another way of saying that homomorphism preserves the $\omega$-product. It preserves also the mixed product, and the semigroup product where applicable:

**Proposition 1.** *Let $\varphi$ be a homomorphism from $(V, \pi)$ to $(\bar{V}, \bar{\pi})$. Let $\bullet$ and $\bar{\bullet}$ be mixed products induced, respectively, by $\pi$ and $\bar{\pi}$. Then:*

*(a) $\varphi(s \bullet v) = s \,\bar{\bullet}\, \varphi(v)$ for all $s \in S$, $v \in V$,*
*(b) $\varphi(s \cdot v) = s \cdot \varphi(v)$ for all $s \in S$, $v \in S \cap V$.*

*Proof.* (a) Both $\pi$ and $\bar{\pi}$ are compatible with $\circ$. As $\bar{\pi} = \pi\varphi$, $\varphi$ is compatible with $\bullet$ by Lemma 1(b). We have $\bullet = \circ_\pi$ and $\bar{\bullet} = \circ_{\pi\varphi}$, so, by Lemma 1(c), $\bar{\bullet}$ is the image of $\bullet$ under $\varphi$. This is exactly the stated property.

(b) Take any $s \in S$ and $v \in S \cap V$. We have $\varphi(v) \in S$.
From (a) and (7) follows: $\varphi(s \cdot v) = \varphi(s \bullet v) = s \,\bar{\bullet}\, \varphi(v) = s \cdot \varphi(v)$. □

**Proposition 2.** *Let* $(V, \pi)$ *be an associative* $\omega$*-product inducing mixed product* $\bullet$*, and* $\bar{V}$ *an arbitrary set. Let* $\varphi : V \to \bar{V}$ *be a surjective mapping compatible with* $\bullet$*. The pair* $(\bar{V}, \pi\varphi)$ *is an* $\omega$*-product on* $\mathbf{S}$ *satisfying (A1) and (A2). It is associative if* $\varphi$ *also satisfies the condition*

$$\varphi(s \bullet v) = s \cdot \varphi(v) \quad \textit{for all } s \in S \textit{ whenever } \varphi(v) \in S. \tag{9}$$

*Proof.* Let $\varphi$ be as stated. Then $\pi\varphi$ is a surjective mapping from $S^{\mathbb{N}}$ to $\bar{V}$, so the pair $(\bar{V}, \pi\varphi)$ is an $\omega$-product on $\mathbf{S}$. Suppose $\mathbf{x} \rhd \mathbf{y}$. From (A1) follows $\pi(\mathbf{x}) = \pi(\mathbf{y})$ and $\varphi(\pi(\mathbf{x})) = \varphi(\pi(\mathbf{y}))$, so $\pi\varphi$ satisfies (A1). The mapping $\pi$ is compatible with $\circ$, and $\varphi$ is compatible with $\bullet = \circ_\pi$. By Lemma 1(b), $\pi\varphi$ is compatible with $\circ$, that is, $\pi\varphi$ satisfies (A2). Condition (9) implies (A3) for $\pi\varphi$: if $\mathbf{x}$ is such that $\varphi(\pi(\mathbf{x})) \in S$, we have, by (5) and (9), $\varphi(\pi(s \circ \mathbf{x})) = \varphi(s \bullet \pi(\mathbf{x})) = s \cdot \varphi(\pi(\mathbf{x}))$. □

# 5 Similar sequences

Define sequences $\mathbf{x} \in S^{\mathbb{N}}$ and $\mathbf{y} \in S^{\mathbb{N}}$ to be *similar*, written $\mathbf{x} \sim \mathbf{y}$, if there exist sequences $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$, $k \geq 1$, such that $\mathbf{z}_1 = \mathbf{x}$, $\mathbf{z}_k = \mathbf{y}$, and $\mathbf{z}_i \rhd \mathbf{z}_{i+1}$ or $\mathbf{z}_i \lhd \mathbf{z}_{i+1}$ for $1 \leq i < k$. In other words, $\sim$ is the reflexive, symmetric and transitive closure of relation $\rhd$. The equivalence classes of $\sim$ are in the following referred to as *similarity classes* of $\mathbf{S}$. The set $S^{\mathbb{N}}/\sim$ of all similarity classes is denoted by $Q$.

**Proposition 3.** *Similarity is consistent with prefixing.*

*Proof.* Take any $s \in S$ and $\mathbf{x} = x_1, x_2, x_3, \dots \in S^{\mathbb{N}}$.
Let $\mathbf{y} = \mathbf{x}|\mathbf{n}$ for some ascending $\mathbf{n} = n_1, n_2, n_3, \dots \in N^{\mathbb{N}}$. By definition of $\mathbf{x}|\mathbf{n}$:

$$\mathbf{y} = (x_1 \cdot \ldots \cdot x_{n_1}),\, (x_{n_1+1} \cdot \ldots \cdot x_{n_2}), \ldots,$$
$$s \circ \mathbf{y} = (s),\, (x_1 \cdot \ldots \cdot x_{n_1}),\, (x_{n_1+1} \cdot \ldots \cdot x_{n_2}), \dots = (s \circ \mathbf{x})|\mathbf{n}',$$

where $\mathbf{n}' = 1, n_1 + 1, n_2 + 1, n_3 + 1, \ldots$ . It follows that $\mathbf{x} \rhd \mathbf{y} \Rightarrow s \circ \mathbf{x} \rhd s \circ \mathbf{y}$. Consider now any $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$ such that $\mathbf{x} \sim \mathbf{y}$. Let $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_k$ be the sequences appearing in the definition of similarity $\mathbf{x} \sim \mathbf{y}$. According to the above, we have $s \circ \mathbf{z}_i \rhd s \circ \mathbf{z}_{i+1}$ or $s \circ \mathbf{z}_i \lhd s \circ \mathbf{z}_{i+1}$ for $1 \leq i < k$, showing that $s \circ \mathbf{x} \sim s \circ \mathbf{y}$. □

As a consequence, $\circ$ has an image under $\mathsf{nat}_\sim$. This image is in the following denoted by $\circ_{\mathrm{Q}}$.

From the definition of $\sim$ follows clearly that (A1) is equivalent to

$$\mathbf{x} \sim \mathbf{y} \;\;\Rightarrow\;\; \pi(\mathbf{x}) = \pi(\mathbf{y}) \;\; \text{for all } \mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}. \tag{10}$$

An associative $\omega$-product is thus fully defined by specifying its value for each similarity class. If this value is different for different similarity classes, we say that the $\omega$-product is *maximal.* From $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \pi(\mathbf{x}) = \pi(\mathbf{y})$ and Proposition 3 follows (A2). A maximal product satisfies thus both (A1) and (A2).

# 6 Free and primary $\omega$-products

The natural mapping $\mathsf{nat}_\sim$ is a surjective mapping from $S^{\mathbb{N}}$ to $Q$, so the pair $(Q, \mathsf{nat}_\sim)$ is an $\omega$-product on $\mathbf{S}$. It is clearly maximal, so it satisfies (A1) and (A2). It satisfies (A3) because $Q \cap S = \varnothing$. The $\omega$-product $(Q, \mathsf{nat}_\sim)$ is thus associative. It induces $\circ_{\mathrm{Q}}$ as the mixed product.

**Proposition 4.** *Each associative* $\omega$*-product on* $\mathbf{S}$ *is a homomorphic image of* $(Q, \mathsf{nat}_\sim)$*.*

*Proof.* Consider any associative $\omega$-product $(V, \pi)$ on $\mathbf{S}$. According to (10), $\pi$ is constant on each class of $\sim$. By a basic property of quotient sets, there exists a mapping $\varphi : Q \to V$ such that $\pi = \mathsf{nat}_\sim\varphi$. This mapping is a homomorphism from $(Q, \mathsf{nat}_\sim)$ to $(V, \pi)$. □

The above property is one of the characteristic properties of free objects. With a somewhat stretched interpretation of sequences in $S^{\mathbb{N}}$ as terms in a term algebra over $\mathbf{S}$, one can say that $(Q, \mathsf{nat}_\sim)$ is "freely generated" by $\mathbf{S}$ with (A1) as the only defining rule. For this reason, $(Q, \mathsf{nat}_\sim)$ is in the following called the *free* $\omega$*-product* on $\mathbf{S}$.

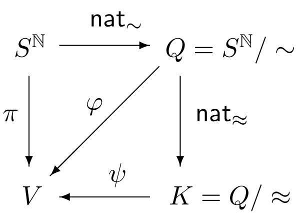
We note that each associative $\omega$-product having the property stated by Proposition 4 is isomorphic to $(Q, \mathsf{nat}_\sim)$. If each associative $\omega$-product is a homomorphic image of $(V, \pi)$, so is, in particular, $(Q, \mathsf{nat}_\sim)$, and these two products are homomorphic images of each other.

Because isomorphic objects are often considered "the same", we loosely refer to any $\omega$-product isomorphic to $(Q, \mathsf{nat}_\sim)$ as "a free $\omega$-product" on $\mathbf{S}$. One can easily see that an $\omega$-product is such a free product if and only if it is maximal and does not assume values from S.

Let $\approx$ be an equivalence on $Q$ compatible with $\circ_{\mathrm{Q}}$. Let $K = Q/\approx$ and $\kappa = \mathsf{nat}_\sim \mathsf{nat}_\approx$. As $\mathsf{nat}_\approx$ is compatible with $\circ_{\mathrm{Q}}$ and $K \cap S = \varnothing$, the pair $(K, \kappa)$ is, according to Proposition 2, an associative $\omega$-product on $\mathbf{S}$. In the following, each such pair $(K, \kappa)$ is called a *primary $\omega$-product* on $\mathbf{S}$.

**Proposition 5.** *Each associative $\omega$-product on $\mathbf{S}$ is a one-to-one homomorphic image of a primary $\omega$-product on $\mathbf{S}$.*

*Proof.* Consider any associative $\omega$-product $(V, \pi)$ on $\mathbf{S}$. Let $\varphi$ be the homomorphism from $(Q, \mathsf{nat}_\sim)$ to $(V, \pi)$ stated by Proposition 4. Define $\approx$ to be the kernel of $\varphi$, that is, $q_1 \approx q_2 \Leftrightarrow \varphi(q_1) = \varphi(q_2)$. Denote $Q/\approx$ by $K$. By a basic property of kernel, there exists a bijection $\psi : K \to V$ such that $\varphi = \mathsf{nat}_\approx \psi$. We have this situation:




As remarked in the proof of Proposition 1(a), $\varphi$ is compatible with $\circ_{\mathrm{Q}}$. This is identical to $\approx$ being compatible with $\circ_{\mathrm{Q}}$. The pair $(K, \mathsf{nat}_\sim \mathsf{nat}_\approx)$ is thus a primary $\omega$-product on $\mathbf{S}$. The bijection $\psi$ is, by our definition, a homomorphism from that $\omega$-product to $(V, \pi)$. $\square$

It is convenient to think of $Q/\approx$ as a partition of $Q$. Proposition 5 shows that all associative $\omega$-products on $\mathbf{S}$ are obtained by assigning different values to classes of a partition of $Q$ consistent with $\circ_{\mathrm{Q}}$. The set $Q$ is fully determined by the semigroup $\mathbf{S}$, and so is the action $\circ_{\mathrm{Q}}$. This latter determines all the consistent partitions of $Q$, and thus (up to the choice of values) all associative $\omega$-products on $\mathbf{S}$.

# 7 Similarity classes in selected cases

## 7.1 Similar sequences in a free semigroup

We recall that a semigroup $(S, \cdot)$ is *free* if it has a subset $G \subset S$ of *generators* such that each element of $S$ can be represented in a unique way as a product of $n \geq 1$ generators.

Let $(S, \cdot)$ be a free semigroup. Given a sequence $\mathbf{x} \in S^{\mathbb{N}}$, represent each element of $\mathbf{x}$ as the unique product of generators: $\mathbf{x} = (g_1 \cdot \ldots \cdot g_{n_1}), (g_{n_1+1} \cdot \ldots \cdot g_{n_2}), (g_{n_2+1} \cdot \ldots \cdot g_{n_3}), \ldots$ . The sequence $\mathbf{g} = g_1, \ldots, g_{n_1}, g_{n_1+1}, \ldots, g_{n_2}, g_{n_2+1}, \ldots, g_{n_3}, \ldots$ of generators appearing in this representation is in the following called the *generating sequence* of $\mathbf{x}$. One can easily see that this sequence $\mathbf{g}$ is unique, that $\mathbf{x}$ is a contraction of $\mathbf{g}$, and that $\mathbf{g}$ is not changed by a contraction of $\mathbf{x}$.

**Proposition 6.** *If $(S, \cdot)$ is a free semigroup, sequences $\mathbf{x} \in S^{\mathbb{N}}$ and $\mathbf{y} \in S^{\mathbb{N}}$ are similar if and only if they have the same generating sequence.*

*Proof.* Suppose $\mathbf{x}$ and $\mathbf{y}$ have the same generating sequence, $\mathbf{g}$. We have then $\mathbf{x} \lhd \mathbf{g} \rhd \mathbf{y}$, so $\mathbf{x} \sim \mathbf{y}$.

Suppose now that $\mathbf{x} \sim \mathbf{y}$. Let $\mathbf{z}_i$, $1 \leq i \leq k$ be the sequences in definition of their similarity. Because generating sequences are unique, and contractions do not change the generating sequence, all of them have the same generating sequence. $\square$

The similarity classes in a free semigroup correspond thus to different sequences of generators.

## 7.2 Similar sequences in a free monoid

A semigroup $(S, \cdot)$ is a *monoid* if it contains a *unit element*, usually denoted by 1, such that $s \cdot 1 = s = 1 \cdot s$ for each $s \in S$. A monoid $(S, \cdot)$ is *free* if it has a subset $G \subset S$ of *generators* such that each element of $S - \{1\}$ can be represented in a unique way as a product of $n \geq 1$ generators, and 1 is not a product of any elements in $S - \{1\}$.

Let $(S, \cdot)$ be a free monoid. Given a sequence $\mathbf{x} \in S^{\mathbb{N}}$, represent each element of $\mathbf{x}$ other than 1 as the unique product of generators. Let $\mathbf{g}$ be the sequence of generators appearing in this representation. If $\mathbf{g}$ is infinite, define the generating sequence of $\mathbf{x}$ to be $\mathbf{g}$. If $\mathbf{g}$ is finite (which is the case when $\mathbf{x}$ ends with $1^{\mathbb{N}}$), define the generating sequence of $\mathbf{x}$ to be $\mathbf{g}$ followed by $1^{\mathbb{N}}$. One can easily see that the generating sequence is unique, that $\mathbf{x}$ is similar to its generating sequence, and that generating sequence is not changed by a contraction of $\mathbf{x}$. The following is proved in the same way as Proposition 6:

**Proposition 7.** *If $(S, \cdot)$ is a free monoid, sequences $\mathbf{x} \in S^{\mathbb{N}}$ and $\mathbf{y} \in S^{\mathbb{N}}$ are similar if and only if they have the same generating sequence.*

The similarity classes in a free monoid correspond thus to different finite or infinite sequences of generators.

## 7.3 Similar sequences in a group

We recall that a monoid $(S, \cdot)$ is a *group* if each element $s \in S$ has the *inverse element* $s^{-1} \in S$ such that $s \cdot s^{-1} = 1 = s^{-1} \cdot s$.

**Proposition 8.** *If $(S, \cdot)$ is a group, all sequences in $S^{\mathbb{N}}$ are similar.*

*Proof.* We show that all sequences are similar to $1^{\mathbb{N}}$.
Consider any sequence $\mathbf{x} = x_1, x_2, x_3, \dots \in S^{\mathbb{N}}$. We have:

$$\begin{aligned}
& x_1, x_2, x_3, \dots = \\
& x_1, (x_1^{-1} \cdot x_1 \cdot x_2), (x_2^{-1} \cdot x_1^{-1} \cdot x_1 \cdot x_2 \cdot x_3), (x_3^{-1} \cdot x_2^{-1} \cdot x_1^{-1} \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_4), \dots \\
& \lhd\, x_1, x_1^{-1}, x_1, x_2, x_2^{-1}, x_1^{-1}, x_1, x_2, x_3, x_3^{-1}, x_2^{-1}, x_1^{-1}, x_1, x_2, x_3, x_4, \dots \\
& \rhd\, (x_1 \cdot x_1^{-1}), (x_1 \cdot x_2 \cdot x_2^{-1} \cdot x_1^{-1}), (x_1 \cdot x_2 \cdot x_3 \cdot x_3^{-1} \cdot x_2^{-1} \cdot x_1^{-1}), \dots \\
& = 1^{\mathbb{N}}.
\end{aligned}$$

□

Any associative $\omega$-product on a group is thus trivial.

## 7.4 Similar sequences in a finite semigroup

A semigroup $(S, \cdot)$ is *finite* if $S$ is a finite set. The fundamental fact about finite semigroups is:

**Proposition 9.** *If $(S, \cdot)$ is a finite semigroup, each sequence in $S^{\mathbb{N}}$ has a contraction of the form $s \circ e^{\mathbb{N}}$ for some $e$ and $s$ in $S$ such that $e \cdot e = e$ and $s \cdot e = s$.*

*Proof.* Consider any sequence $\mathbf{x} \in S^{\mathbb{N}}$. For each $s \in S$, let $C(s)$ be the set of all pairs $(p, q)$ of natural numbers $p < q$ such that $\mathbf{x}(p+1) \cdot \ldots \cdot \mathbf{x}(q) = s$. If $S$ is finite, the classes $C(s)$ constitute a finite partition of the set of all pairs of natural numbers. By a combinatorial result of Ramsey (Theorem A in [15]), there exists an infinite subset $\mathbb{N}_e \subset \mathbb{N}$ such that all pairs $(p, q)$ where $p, q \in \mathbb{N}_e$ belong to the same class $C(e)$ for some $e \in S$.

Let $\mathbf{n} = n_1, n_2, n_3, \dots$ consist of all elements of $\mathbb{N}_e$ in ascending order. Let $\mathbf{y} = \mathbf{x}|\mathbf{n}$. By the choice of $\mathbf{n}$, we have $\mathbf{y}(i) = \mathbf{x}(n_{i-1} + 1) \cdot \ldots \cdot \mathbf{x}(n_i) = e$ for all $i > 1$. Denoting $\mathbf{y}(1)$ by $x$, we have $\mathbf{x} \rhd x, e, e, e, \dots$ .

By the choice of $\mathbf{n}$, we have also $e = \mathbf{x}(n_1 + 1) \cdot \ldots \cdot \mathbf{x}(n_2) \cdot \mathbf{x}(n_2 + 1) \cdot \ldots \cdot \mathbf{x}(n_3) = e \cdot e$.

If $x, e, e, e, \dots$ is a contraction of $\mathbf{x}$, so is the sequence $(x \cdot e), e, e, e, \dots$ . Denoting $x \cdot e$ by $s$, we have $\mathbf{x} \rhd s, e, e, e, \dots$, where $s \cdot e = x \cdot e \cdot e = x \cdot e = s$. □

(This fact has been exploited in practically all work with automata on infinite words. It seems to appear for the first time in 1960 in a paper by Büchi [3]. Lemma 1 in that paper states that an infinite word can be divided into finite words belonging to the seme class. It is proved there using Ramsey's theorem; the idea is credited to a discussion with J. B. Wright. All subsequent formulations are in terms of a homomorphism from an infinite word into a finite semigroup. The form $s \circ e^{\mathbb{N}}$ seems to appear for the first time in 1973 in a paper by Schützenberger [18]. It is proved there by induction on the number of different values of products of partial sequences $\mathbf{x}(i) \cdot \ldots \cdot \mathbf{x}(j)$. A proof by induction on the number of different elements $\mathbf{x}(i)$ can be found in [9]. A non-inductive proof was given in 1981 by Thomas [22]; it appears also in [10] and [12].)

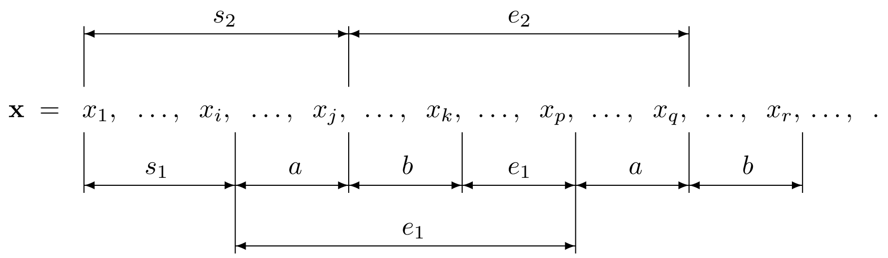
An element $e \in S$ such that $e \cdot e = e$ is called an *idempotent.* A pair $(s, e)$ with idempotent $e$ and $s \cdot e = s$ is called a *linked pair.* We write $\mathbf{x} \rhd (s, e)$ to mean $(s, e)$ is a linked pair such that $\mathbf{x} \rhd s \circ e^{\mathbb{N}}$.

For a semigroup $(S, \cdot)$, let $S^1$ denote the set $S$ if $(S, \cdot)$ is a monoid. Otherwise, let $S^1 = S \cup \{1\}$ where $s \cdot 1 = s = 1 \cdot s$ for each $s \in S$. Linked pairs $(s_1, e_1)$ and $(s_2, e_2)$ are said to be *conjugated* if there exist $x, y \in S^1$ such that $e_1 = x \cdot y$, $e_2 = y \cdot x$, $s_2 = s_1 \cdot x$.

One can easily verify that conjugation is an equivalence relation in the set of linked pairs. The equivalence classes of that relation are in the following referred to as *conjugation classes.*

**Proposition 10.** *Let $(s_1, e_1)$ and $(s_2, e_2)$ be two linked pairs. If $(s_1, e_1) \lhd \mathbf{x} \rhd (s_2, e_2)$ for some $\mathbf{x} \in S^{\mathbb{N}}$, $(s_1, e_1)$ and $(s_2, e_2)$ are conjugated.*

*Proof.* Let $\mathbf{x} = x_1, x_2, x_3, \ldots$ . Suppose $(\mathbf{x}|\mathbf{n}_1) = s_1 \circ e_1^{\mathbb{N}}$ and $(\mathbf{x}|\mathbf{n}_2) = s_2 \circ e_2^{\mathbb{N}}$ for some $\mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^{\mathbb{N}}$. Consider triples $i < j < k$ such that $i$ and $k$ are in $\mathbf{n}_1$, and $j$ is in $\mathbf{n}_2$. Associate with each such triple a pair $(a, b) \in S \times S$ where $a = x_{i+1} \cdot \ldots \cdot x_j$ and $b = x_{j+1} \cdot \ldots \cdot x_k$. There are infinitely many such triples that are disjoint, that is, the elements of one triple are either all greater than, or all less than, all elements of another. Because $S$ is finite, there are only finitely many different pairs $(a, b)$. Hence, at least one such pair must be associated with at least two disjoint triples, and there must exist $i < j < k < p < q < r$ such that $i, k, p, r$ are in $\mathbf{n}_1$, $j, q$ are in $\mathbf{n}_2$, and $x_{i+1} \cdot \ldots \cdot x_j = x_{p+1} \cdot \ldots \cdot x_q = a$, $x_{j+1} \cdot \ldots \cdot x_k = x_{q+1} \cdot \ldots \cdot x_r = b$. We have this situation:




It follows that:

$$
\begin{aligned}
x_{i+1} \cdot \ldots \cdot x_p &= e_1 = a \cdot b \cdot e_1, \\
x_{j+1} \cdot \ldots \cdot x_q &= e_2 = b \cdot e_1 \cdot a, \\
x_1 \cdot \ldots \cdot x_j &= s_2 = s_1 \cdot a.
\end{aligned}
$$

Denoting $a = x$, $b \cdot e_1 = y$, we have $s_2 = s_1 \cdot x$, $e_1 = x \cdot y$, $e_2 = y \cdot x$. □

(The earliest appearance of terms "linked pair" and "conjugated pairs" in a generally available publication seems to be in [24]. Proposition 10, stated as a property of homomorphisms, appears in [25], where it is credited to Ph.D. thesis of J-P. Pécuchet from 1987. The proof given here is a slightly simplified version of proof from [12].)

**Proposition 11.** *If $(S, \cdot)$ is a finite semigroup, sequences $\mathbf{x} \in S^{\mathbb{N}}$ and $\mathbf{y} \in S^{\mathbb{N}}$ are similar if and only if $\mathbf{x} \rhd (s_1, e_1)$ and $\mathbf{y} \rhd (s_2, e_2)$ for linked pairs $(s_1, e_1)$ and $(s_2, e_2)$ in the same conjugation class.*

*Proof.* Consider any $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$. By Proposition 9, $\mathbf{x} \rhd (s_1, e_1)$ and $\mathbf{y} \rhd (s_2, e_2)$ for some linked pairs $(s_1, e_1)$ and $(s_2, e_2)$. Suppose these linked pairs are conjugated. Let $x, y$ be those in the definition of their conjugation. We have then

$$\begin{aligned}
&\mathbf{x} = s_1, e_1, e_1, e_1, \dots = (s_2 \cdot y), (x \cdot y), (x \cdot y), \dots \lhd \\
&s_2, y, x, y, x, y, \dots \rhd \\
&(s_2), (y \cdot x), (y \cdot x), (y \cdot x), \dots = s_2, e_2, e_2, e_2, \dots = \mathbf{y} \ ,
\end{aligned}$$

showing that $\mathbf{x} \sim \mathbf{y}$.

Suppose now $\mathbf{x} \sim \mathbf{y}$. Let $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k$ be the sequences in the definition of their similarity. A sequence of the form $s \circ e^{\mathbb{N}}$ can only be contracted to itself; with $\rhd$ being transitive, we must have

$$(s_1, e_1) \lhd \mathbf{z}'_1 \rhd \mathbf{z}'_2 \lhd \dots \rhd \mathbf{z}_{n-1} \lhd \mathbf{z}'_n \rhd (s_2, e_2)$$

where sequences $\mathbf{z}'_1, \dots, \mathbf{z}'_n \in S^{\mathbb{N}}$ are some, or all, of $\mathbf{z}_1, \dots, \mathbf{z}_n$. Using Proposition 9 and transitivity of $\rhd$, we can replace every second $\mathbf{z}'_i$ by a linked pair $(s'_i, e'_i)$. The stated result follows from Proposition 10. □

Similarity classes in a finite semigroup correspond thus to conjugation classes of linked pairs. The number of distinct linked pairs, and thus the number of conjugation classes, cannot exceed $mn$, where $m$ is the number of elements of $S$, and $n$ is the number of idempotents of $S$. Consequently, an associative $\omega$-product on a finite semigroup can assume at most $mn$ distinct values and is finitely specified.

# 8 Infinite product defined as limit

As was indicated at the beginning, an $\omega$-product $x_1 \cdot x_2 \cdot x_3 \cdot \dots$ need not be the limit of partial products $x_1 \cdot x_2 \cdot \dots \cdot x_n$ for $n \to \infty$. However, it may be so. Suppose $S$ is a subset of topological space $T$ that defines the notions of convergence and limit. The limit of a convergent sequence $\mathbf{x} \in T^{\mathbb{N}}$ is in the following denoted by $\mathsf{lim}(\mathbf{x})$. (We assume that $T$ is a Hausdorff space, implying that each sequence has at most one limit.)

For a sequence $\mathbf{x} = x_1, x_2, x_3, \dots \in S^{\mathbb{N}}$, let $\widehat{\mathbf{x}}$ denote the sequence of its partial products, that is, $\widehat{\mathbf{x}}(i) = x_1 \cdot x_2 \cdot \dots \cdot x_i$ for $i \geq 1$. Suppose $\widehat{\mathbf{x}}$ is convergent for each $\mathbf{x} \in S^{\mathbb{N}}$. For each such $\mathbf{x}$, define $\pi(\mathbf{x}) = \mathsf{lim}(\widehat{\mathbf{x}})$. The pair $(V, \pi)$ where $V = \{\, \mathsf{lim}(\widehat{\mathbf{x}}) \mid \mathbf{x} \in S^{\mathbb{N}}\}$ is an $\omega$-product on $\mathbf{S}$.

This product satisfies (A1): if $\mathbf{x} \rhd \mathbf{y}$, the sequence of partial products of $\mathbf{y}$ is a sub-sequence of $\widehat{\mathbf{x}}$, and converges to the same limit as $\widehat{\mathbf{x}}$.

Conditions (A2) and (A3) translate into a condition that the semigroup product has an extension to action $\cdot : S \times T \to T$, continuous in the sense that for each convergent $\mathbf{x} = x_1, x_2, x_3, \dots \in S^{\mathbb{N}}$, the sequence $s \cdot \mathbf{x} = s \cdot x_1,\ s \cdot x_2,\ s \cdot x_3, \dots$ converges to $s \cdot \mathsf{lim}(\mathbf{x})$. This extension restricted to $S \times V$ is the mixed product induced by $\pi$.

# 9 Infinite product in presence of left zeros

An element $z \in S$ is a *left zero* if $z \cdot s = z$ for all $s \in S$. The set of left zeros of $\mathbf{S}$ is in the following denoted by $Z$. We note that if $z$ is a left zero, so is $s \cdot z$ for all $s \in S$: $(s \cdot z) \cdot t = s \cdot (z \cdot t) = s \cdot z$.

Let $\mathbf{x} = x_1, x_2, x_3, \dots \in S^{\mathbb{N}}$ contain a left zero, that is, $x_p \in Z$ for some $p \geq 1$. Then, the partial product $x_1 \cdot x_2 \cdot \dots \cdot x_n$ for each $n \geq p$ is equal to $x_1 \cdot x_2 \cdot \dots \cdot x_p \in Z$. This value is in the following denoted by $\zeta(\mathbf{x})$. Note that it does not depend on $p$ being the smallest one with $x_p \in Z$.

Let $\mathbf{y} = y_1, y_2, y_3, \dots$ be any contraction of $\mathbf{x}$. It must contain an element $y_q = x_i \cdot \dots \cdot x_p \cdot \dots \cdot x_j$, $i \leq p \leq j$, which is a left zero. One can easily see that each partial product $y_1 \cdot y_2 \cdot \dots \cdot y_n$ with $n \geq q$ is equal to $\zeta(\mathbf{x})$, meaning that $\mathbf{x} \rhd \mathbf{y} \Rightarrow \zeta(\mathbf{x}) = \zeta(\mathbf{y})$.

Suppose $(V, \pi)$ is an associative $\omega$-product on $\mathbf{S}$ with $V \subseteq S$. Let $\mathbf{x}$ be as before. From (A3) follows then $\pi(\mathbf{x}) = (x_1 \cdot \dots \cdot x_p) \cdot (x_{p+1} \cdot x_{p+2} \cdot \dots) = \zeta(\mathbf{x})$. That means, $\pi(\mathbf{x})$ is in such case uniquely defined as $\zeta(\mathbf{x})$ for each $\mathbf{x} \in S^{\mathbb{N}}$ containing left zero.

**Proposition 12.** *Let $(V, \pi)$ be any $\omega$-product on $\mathbf{S}$ (not necessarily with $V \subseteq S$) such that $\pi(\mathbf{x}) = \zeta(\mathbf{x})$ for each $\mathbf{x} \in S^{\mathbb{N}}$ that contains left zero, and (A1)–(A3) hold for all sequences in $S^{\mathbb{N}}$ that do not contain left zero. The product is associative.*

*Proof.* As noted before, $\mathbf{x} \rhd \mathbf{y} \Rightarrow \zeta(\mathbf{x}) = \zeta(\mathbf{y})$ for $\mathbf{x}$ containing left zero, so (A1) holds for all such sequences. It holds for the remaining sequences by assumption.

We have further $\pi(\mathbf{x}) \in S$ and $\pi(s \circ \mathbf{x}) = s \cdot \zeta(\mathbf{x}) = s \cdot \pi(\mathbf{x})$ for $\mathbf{x}$ containing left zero, so (A3) holds for all such sequences. It holds for the remaining sequences by assumption.

Take any $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$ such that $\pi(\mathbf{x}) = \pi(\mathbf{y})$. If any of $\mathbf{x}, \mathbf{y}$ contains left zero, both products have value in $S$ and (A2) is implied by (A3). If none of them contains left zero, (A2) holds by assumption. □

# 10 Examples

**Example 1.** (Finite case). Let $\mathbf{S} = (\{0, 1\}, \cdot)$, where $\cdot$ is multiplication of integers. Both elements of $\mathbf{S}$ are idempotents, so there are four pairs $(s, e)$. One of them, $(1, 0)$, is not a linked pair because $1 \cdot 0 \neq 1$. The remaining three are linked pairs, and none of them are conjugated, so $\mathbf{S}$ has three similarity classes, corresponding to the pairs $(0, 0)$, $(0, 1)$, $(1, 1)$. Let these pairs denote the corresponding classes. The class $(0, 0)$ consists of all sequences with infinitely many zeros. The class $(0, 1)$ consists of all sequences with finitely many zeros, and the class $(1, 1)$ contains only the sequence $1^{\mathbb{N}}$. The action $\circ_{\mathrm{Q}}$ and all the partitions of $Q$ consistent with it are:

| $\circ_{\mathrm{Q}}$ | $(0,0)$ | $(0,1)$ | $(1,1)$ |
|---|---|---|---|
| 0 | $(0,0)$ | $(0,1)$ | $(0,1)$ |
| 1 | $(0,0)$ | $(0,1)$ | $(1,1)$ |

| | | |
|---|---|---|
| $(0,0)$ | $(0,1)$ | $(1,1)$ |
| $(0,0)$ | $(0,1)$ $(1,1)$ | |
| $(0,0)$ $(0,1)$ | $(1,1)$ | |
| $(0,0)$ $(0,1)$ $(1,1)$ | | |

For $\mathbf{x} \in \{0, 1\}^{\mathbb{N}}$, define $\pi(\mathbf{x}) = a$ if $\mathbf{x}$ contains infinitely many zeros, $\pi(\mathbf{x}) = 0$ if $\mathbf{x}$ contains finitely many zeros, and $\pi(\mathbf{x}) = 1$ if $\mathbf{x}$ does not contain zeros. The pair $(\{a, 0, 1\}, \pi)$ is an $\omega$-product on $\mathbf{S}$. The three values of $\pi$ correspond to the similarity classes of $\mathbf{S}$, so the product is maximal and satisfies (A1), (A2). To verify (A3), replace $(0, 1)$ and $(1, 1)$ in the table for $\circ_{\mathrm{Q}}$ by 0 and 1, respectively. The subtable corresponding to them becomes a correct multiplication table for the semigroup $\mathbf{S}$. The $\omega$-product $(\{a, 0, 1\}, \pi)$ is thus associative.

Define now $\pi'(\mathbf{x}) = a$ if $\mathbf{x}$ contains infinitely many zeros, and $\pi'(\mathbf{x}) = b$ otherwise. The pair $(\{a, b\}, \pi')$ is an $\omega$-product on $\mathbf{S}$. It has value $a$ on the similarity class $(0, 0)$, and $b$ on the similarity classes $(0, 1)$ and $(1, 1)$. The partition of $Q$ into $\{(0, 0)\}$ and $\{(0, 1), (1, 1)\}$ is consistent with $\circ_{\mathrm{Q}}$, so $\pi'$ satisfies (A1) and (A2). It satisfies (A3) because of $\{0, 1\} \cap \{a, b\} = \varnothing$. The $\omega$-product $(\{a, b\}, \pi')$ is associative.

**Example 2.** (Finite case with conjugated pairs). Let $\mathbf{S} = (\{a, b\}, \cdot)$, where $a \cdot a = a \cdot b = a$ and $b \cdot a = b \cdot b = b$. Both elements of $\mathbf{S}$ are idempotents, so there are four pairs $(s, e)$: $(a, a)$, $(a, b)$, $(b, a)$, $(b, b)$. All of them are linked pairs; $(a, a)$ is conjugated with $(a, b)$, and $(b, a)$ is conjugated with $(b, b)$. The semigroup $\mathbf{S}$ has thus two similarity classes, corresponding to pairs $(a, a)$ and $(b, b)$. They consist of sequences starting, respectively, with $a$ and $b$.

For $\mathbf{x} \in \{a, b\}^{\mathbb{N}}$, define $\pi(\mathbf{x}) = a$ if $\mathbf{x}$ starts with $a$, and $\pi(\mathbf{x}) = b$ if $\mathbf{x}$ starts with $b$. This assignment is a bijection from $Q$, and (A3) is easily verified. The product is associative.

**Example 3.** (Concatenation of words from $A^+$). Let $A$ be a finite alphabet, $A^+$ the set of all finite nonempty words over $A$, and $A^\omega$ the set of all infinite words over $A$. Let $\mathbf{S} = (A^+, \cdot)$, where $\cdot$ is concatenation of words. For $\mathbf{x} = w_1, w_2, w_3, \ldots \in (A^+)^{\mathbb{N}}$, define $\pi(\mathbf{x})$ to be the sequence of letters appearing in the words $w_1, w_2, w_3, \ldots$ . The pair $(A^\omega, \pi)$ is an $\omega$-product on $\mathbf{S}$.

$\mathbf{S}$ is a free semigroup with $A$ as the set of generators. As found in Section 7.1, similarity classes of $\mathbf{S}$ correspond to different generating sequences. The value of $\pi(w_1, w_2, w_3, \ldots)$ is just the generating sequence of $w_1, w_2, w_3, \ldots$; the product $\pi$ is thus maximal. It is free because $A^\omega \cap A^* = \varnothing$.

An equivalence on $A^\omega$ compatible with $\circ$ is often referred to as a left congruence on $A^\omega$. The primary $\omega$-products $(K, \kappa)$ on $\mathbf{S} = (A^+, \cdot)$ correspond thus to different left congruences on $A^\omega$, the members of $K$ being the congruence classes. Left congruences are related to so-called finite-state $\omega$-languages (cf. [20]). As shown in [8], a subset of $A^\omega$ is finite-state if and only if it is a union of classes of a left congruence with a finite index. Finite-state languages correspond thus to subsets of finite primary products on $(A^+, \cdot)$.

**Example 4.** (Free concatenation of words from $A^*$). Let $A$ be a finite alphabet, and $A^*$ the set of all finite words over $A$, including the empty word $\varepsilon$. Let $A^\omega$ be the set of all infinite words over $A$. Let $\mathbf{S} = (A^*, \cdot)$, where $\cdot$ is concatenation of words.

Let $\mathbf{x} = w_1, w_2, w_3, \ldots$ be any sequence of words from from $A^*$. If infinitely many among the words $w_i$ are nonempty, define $\pi(\mathbf{x})$ to be the sequence of letters appearing in the words $w_1, w_2, w_3, \ldots$. This sequence is an infinite word. If only finitely many words $w_i$ are nonempty, the sequence of letters appearing in in $w_1, w_2, w_3, \ldots$ is finite, and possibly empty. In this case, define $\pi(\mathbf{x})$ to be that sequence of letters followed by $\lambda$, where $\lambda$ is a new symbol not in $A$. The result is a word of the form $w\lambda$ where $w \in A^*$. Denote the set of all such words by $A^*\lambda$. The pair $(A^*\lambda \cup A^\omega, \pi)$ is an $\omega$-product on $\mathbf{S}$.

The semigroup $\mathbf{S}$ is a free monoid with set of generators $A$ and unit $\varepsilon$. As found in Section 7.2, similarity classes of $\mathbf{S}$ correspond to different generating sequences. The values of $\pi(w_1, w_2, w_3, \ldots)$ in $A^\omega$ are just the generating sequences of $w_1, w_2, w_3, \ldots$ . The values in $A^*\lambda$ are in a one-to-one correspondence with the generating sequences of $w_1, w_2, w_3, \ldots$ . The product $\pi$ is thus maximal. It is free because $(A^*\lambda \cup A^\omega) \cap A^* = \varnothing$.

**Example 5.** (Natural concatenation of words from $A^*$). Let $A$, $A^*$, $A^\omega$, and $\mathbf{S}$ be as in Example 4. For $\mathbf{x} = w_1, w_2, w_3, \ldots \in (A^*)^{\mathbb{N}}$, define $\pi(\mathbf{x})$ to be the sequence of letters appearing in the words $w_1, w_2, w_3, \ldots$ . This sequence is finite or not, depending on how many among the words $w_i$ are nonempty. The pair $(A^* \cup A^\omega, \pi)$ is an $\omega$-product on $\mathbf{S}$.

The product $\pi$ is maximal for the same reason as in Example 4. It is not free because it assumes values from $A^*$. One can easily see that $w \cdot \pi(w_1, w_2, w_3, \ldots) = \pi(w, w_1, w_2, w_3, \ldots)$ for all $w \in A^*$. That means $\pi$ satisfies (A3) and is associative.

The mapping $\pi$ can also be defined in the way discussed in Section 8, as the limit of $w_1 \cdot w_2 \cdot \ldots \cdot w_i$ for $i \to \infty$. The topology used for this purpose may be that introduced in [16], where limit is defined as the least upper bound of a sequence of words ordered by the relation of being a prefix. It may also be a topology defined by a suitable metric on the set $A^* \cup A^\omega$. As the $\omega$-product from Example 3 can be defined in the same way, the product defined here appears as a natural extension of that product.

We note that $(A^* \cup A^\omega, \pi)$ is a homomorphic image of $(A^*\lambda \cup A^\omega, \pi)$ from Example 4 under homomorphism $\varphi$ defined by $\varphi(w\lambda) = w$ for $w \in A^*\lambda$ and $\varphi(w) = w$ for $w \in A^\omega$. Although $\varphi$ is a bijection, the two products are not isomorphic because $(A^*\lambda \cup A^\omega) \cap A^* = \varnothing$ while $(A^* \cup A^\omega) \cap A^* = A^*$.

**Example 6.** (Infinite concatenation of processes). Let $A$, $A^*$, $A^*\lambda$, and $A^\omega$ be as in Example 4. Define $A^{*\omega} = A^*\lambda \cup A^\omega$ and $A^\infty = A^* \cup A^*\lambda \cup A^\omega$.

Let the words in $A^\infty$ represent runs of a process. A run that terminates after outputting a finite sequence of symbols from $A$ is represented by that sequence of symbols: a word from $A^*$. A run that never terminates and keeps outputting new symbols is represented by the sequence thus produced: a word from $A^\omega$. A run that produces a finite sequence $w$ of symbols, and then goes indefinitely on without producing more output, is represented by the word $w\lambda$. The *behavior* of a process is the set $X \subseteq A^\infty$ of sequences representing all possible runs of the process.

For words $x, y \in A^\infty$, define:

$$x \cdot y = \begin{cases} xy & \text{if } x \in A^*, \\ x & \text{otherwise,} \end{cases}$$

where $xy$ is $x$ followed by $y$. This operation describes a run of two processes arranged so that termination of the first activates the second; $x$ represents output of the first process, and $y$ output of the second (if ever started). The operation is associative, so $\mathbf{S} = (A^\infty, \cdot)$ is a semigroup. The words in $A^{*\omega}$ are left zeros of $\mathbf{S}$.

Let $\mathbf{x} = w_1, w_2, w_3, \ldots$ be a sequence of words from $A^\infty$. If all words $w_i$ are in $A^*$, define $\pi(\mathbf{x})$ in the same way as in Example 4. If $w_k \in A^{*\omega}$ for some $k \geq 1$, define $\pi(\mathbf{x}) = w_1 \cdot \ldots \cdot w_k$.

The pair $(A^{*\omega}, \pi)$ is an $\omega$-product on $\mathbf{S}$. It is associative by Proposition 12.

The extension of $\cdot$ to subsets of $A^\infty$ represents behavior of two processes where termination of the first activates the second. If $X$ is behavior of the first process and $Y$ of the second, behavior of the composite process is $X \cdot Y$ (under assumption that processes do not communicate otherwise). The extension of $\pi$ to subsets represents behavior of a sequence of processes where termination of one triggers the next: if $X_i$ is behavior of the $i$-th process, $\pi(X_1, X_2, X_3, \ldots)$ is behavior of the composite process.

**Example 7.** (Infinite Therien product). The product $X \cdot Y$ from Example 6 does not correctly describe the concatenation of processes if the set $Y$ may be empty (meaning the second process has no valid runs). Namely, $X \cdot \varnothing = \varnothing$, even if $X$ contains infinite runs where the control never reaches the second process. The same applies to the $\omega$-product, which becomes empty if any of $X_i$ is empty.

In [21], Therien suggested a new operation on subsets $X, Y \subseteq A^\infty$ that correctly describes the concatenation in presence of empty behavior. This operation, adapted to our example, is:

$$X {}^\frown Y = \begin{cases} X \cdot Y & \text{if } Y \neq \varnothing, \\ X \cap A^{*\,\omega} & \text{otherwise,} \end{cases}$$

where $X \cdot Y$ is as in Example 6. Using the definition of $\cdot$, one can verify that ${}^\frown$ is associative, so $\mathbf{S} = (2^{A^\infty}, {}^\frown)$ is a semigroup, with $\varnothing$ as one of its left zeros.

Let $\mathbf{X} = X_1, X_2, X_3, \dots$ be a sequence of subsets of $A^\infty$. If none of $X_i$ is empty, define $\widehat{\pi}(\mathbf{X})$ to be the same as $\pi(\mathbf{X})$ in Example 6. If $X_k = \varnothing$ for some $k$, define $\widehat{\pi}(\mathbf{x}) = X_1{}^\frown \dots {}^\frown X_k$.

The pair $(2^{A^{*\,\omega}}, \widehat{\pi})$ is an $\omega$-product on $\mathbf{S}$ that correctly describes behavior of a sequential composition of processes with behaviors $X_1, X_2, X_3, \dots$ that may be empty. This product is associative by Proposition 12.

**Example 8.** (Infinite series). Let $\mathbf{S} = (\mathbb{R}_+, +)$, where $\mathbb{R}_+$ is the set of all nonnegative real numbers, and $+$ is addition. For a sequence $\mathbf{x} = r_1, r_2, r_3, \dots \in \mathbb{R}_+^{\mathbb{N}}$, define $\pi(\mathbf{x})$ to be the sum of the series $r_1 + r_2 + r_3 + \dots$ if the series converges, or $\infty$ otherwise.

The pair $(\mathbb{R}_+ \cup \{\infty\},\ \pi)$ is an $\omega$-product on $\mathbf{S}$. It is defined as limit of partial sums in the topology of real numbers supplemented by $\infty$ as the limit of sequences that increase without a bound. Let "+" be extended by defining $r + \infty = \infty$. With this extension, we have $\mathsf{lim}(r + r_1, r + r_2, r + r_3, \dots) = r + \mathsf{lim}(r_1, r_2, r_3, \dots)$ for all convergent sequences $r_1, r_2, r_3, \dots \in \mathbb{R}_+^{\mathbb{N}}$. The $\omega$-product $(\mathbb{R}_+ \cup \{\infty\},\ \pi)$ is thus associative.

If $\mathbb{R}_+$ is replaced by the set $\mathbb{R}$ of all real numbers, some sequences of partial sums do not converge. One might hope to extend the topology in some way to make them convergent. But $\{\mathbb{R}, +\}$ is a group, and has, by Proposition 8, only one similarity class. That means one cannot define a useful associative sum for series on $\{\mathbb{R}, +\}$ in this, or any other, way.

# 11 Additional topics

## 11.1 Independence of axioms

It is natural to ask if axioms (A1)–(A3) are independent. The three examples below attempt to answer this question. The $\omega$-product in Example 9 satisfies (A2) and (A3), but not (A1); therefore, (A1) can not be implied by them. Similarly, the product in Example 10 satisfies (A1) and (A3), but not (A2); the product in Example 11 satisfies (A1) and (A2) but not (A3). The axioms (A1)–(A3) are thus in a general sense independent of each other. However, if $S \cap V$ is not empty, (A3) implies (A2) whenever $\pi(\mathbf{x})$ and $\pi(\mathbf{y})$ are in $S$.

**Example 9.** Let $\mathbf{S} = (\{0,1\}, \cdot)$, be the semigroup from Example 1. For a sequence $\mathbf{x} \in \{0,1\}^{\mathbb{N}}$, define $\pi(0^{\mathbb{N}}) = a$, $\pi(1^{\mathbb{N}}) = 1$, otherwise $\pi(\mathbf{x}) = 0$. The $\omega$-product $(\{0,1,a\},\ \pi)$ does not satisfy (A1) because $\pi(1,0,0,\dots) \neq \pi((1 \cdot 0), 0, 0, \dots)$. It satisfies (A2) and induces this mixed product: $0 \bullet a = a,\ 0 \bullet 0 = 0 \bullet 1 = 1 \bullet 0 = 0,\ 1 \bullet 1 = 1$. We have $s \bullet v = s \cdot v$ for all $s, v \in \{0,1\}$, so $\pi$ satisfies (A3) as well.

**Example 10.** Let $\mathbf{S} = (\{0,1\}, \cdot)$, be the semigroup from Example 1. For a sequence $\mathbf{x} \in \{0,1\}^{\mathbb{N}}$, define $\pi(\mathbf{x}) = a$ if $\mathbf{x}$ is in one of the classes $(0,0), (1,1)$; otherwise $\pi(\mathbf{x}) = b$. By its construction, the $\omega$-product $(\{a,b\},\ \pi)$ satisfies (A1). It also satisfies (A3) because $S \cap V = \varnothing$. However, it does not satisfy (A2) because $\pi(0^{\mathbb{N}}) = \pi(1^{\mathbb{N}})$ while $\pi(0 \circ 0^{\mathbb{N}}) \neq \pi(0 \circ 1^{\mathbb{N}})$.

**Example 11.** Let $\mathbf{S} = (\{0,1\}, \cdot)$, be the semigroup from Example 1. For a sequence $\mathbf{x} \in \{0,1\}^{\mathbb{N}}$, define $\pi(\mathbf{x}) = 1$ if $\mathbf{x}$ is in one of the classes $(0,0), (0,1)$; otherwise $\pi(\mathbf{x}) = 0$. The partition of the set of similarity classes into $\{(0,0), (0,1)\}$ and $\{(1,1)\}$ is compatible with $\circ_{\mathrm{Q}}$, so the $\omega$-product $(\{0,1\},\ \pi)$ satisfies (A1) and (A2). However, it does not satisfy (A3) because $0 \cdot \pi(1^{\mathbb{N}}) = 0$ while $\pi(0 \circ 1^{\mathbb{N}}) = 1$.

### 11.2 Mixed product does not define $\omega$-product

If the $\omega$-product defines mixed product, one might ask if the opposite is true. We might hope to define the $\omega$-product in terms of mixed product, for example, by means of infinite factorization

$$\pi(x_1, x_2, x_3, \dots) = (x_1 \bullet (x_2 \bullet (x_3 \bullet (\dots)))).$$

The following example shows that different associative $\omega$-products may induce the same mixed product.

**Example 12.** Let $\mathbf{S} = (\{0, 1, 2\}, \cdot)$, where $s_1 \cdot s_2 = \min(s_1, s_2)$.
$\mathbf{S}$ has six similarity classes: $(0,0)$, $(0,1)$, $(0,2)$, $(1,1)$, $(1,2)$, $(2,2)$. For a sequence $\mathbf{x} \in \{0,1,2\}^{\mathbb{N}}$, define:

$$\pi'(\mathbf{x}) = a \text{ if } \mathbf{x} \text{ is in one of the classes } (1,1), (1,2), (2,2); \quad \text{otherwise } \pi'(\mathbf{x}) = b.$$
$$\pi''(\mathbf{x}) = a \text{ if } \mathbf{x} \text{ is in one of the classes } (1,2), (2,2); \quad \text{otherwise } \pi''(\mathbf{x}) = b.$$

The mappings $\pi'$ and $\pi''$ correspond to these partitions of $Q$:

$$\{(0,0), (0,1), (0,2)\} \quad \{(1,1), (1,2), (2,2)\};$$
$$\{(0,0), (0,1), (0,2), (1,1)\} \quad \{(1,2), (2,2)\}.$$

One can easily verify that both partitions are compatible with $\circ_{\mathrm{Q}}$. Because $\{0,1,2\} \cap \{a,b\} = \varnothing$, both $\omega$-products $(\{a,b\}, \pi')$ and $(\{a,b\}, \pi'')$ are associative. They induce the same mixed product, namely:

$$1 \bullet a = 2 \bullet a = a, \quad 0 \bullet a = 0 \bullet b = 1 \bullet b = 2 \bullet b = b.$$

## Acknowledgements

The author is indebted to Antoni Mazurkiewicz and two anonymous referees for a number of valuable suggestions.

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