

# Associative Omega-product of Processes\*

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## Abstract

The notion of an associative omega-product is applied to processes. Processes are one of the ways to represent behavior of Petri nets. They have been studied for some years as an alternative to traces and dependence graphs. One advantage of processes, as compared to traces, is a very simple way to define infinite concatenation. We take a closer look at this operation, and show that it is a free associative omega-product of finite processes. Its associativity simplifies some arguments about infinite concatenation, as illustrated by the proof of interleaving theorem.

## 1 Introduction

The analysis of systems that do not stop is naturally reduced to considering their infinite behavior. A common approach is to consider infinite composition of steps that make that behavior. One arrives thus at infinite composition of words, traces, processes, state transformations, or other objects. This infinite composition is not an infinite repetition of a finite operation, but a new operation on an infinite sequence of operands. In the following, such operation is referred to as an  $\omega$ -product, and its operands as factors.

Some  $\omega$ -products are associative, that is, their result does not change if neighboring factors are combined before applying the infinite operation; some are not. For a long time, infinite composition and its associativity have been exploited in a rather informal way. Apparently, the first formal definition of an associative  $\omega$ -product was published in [13]. Slightly before, the present author proposed a set of axioms for an associative  $\omega$ -product in a Dagstuhl Seminar lecture, of which only an abstract [15] was published. Being applied to finite automata, the infinite product was mainly studied for finite semigroups. An extensive review can be found in [14]. A recent paper [16] by the present author is a general study of associative  $\omega$ -products for arbitrary semigroups.

In this paper, we apply some results from [16] to "processes" introduced in [1–5]. These processes are slightly different from those in earlier publications, such as [6, 7]. The main idea is to describe evolution of a Petri net by a composition of certain "atomic processes". This is analogous to the way sequential systems are described by words composed of letters. As expressed in [2], processes are "non-linear words". These "words" can be concatenated in various ways to define "languages" of processes.

We begin, in Section 2, by recalling the necessary definitions and results from [16]. Section 3 is an informal introduction to processes and their concatenation. Processes are sets of certain "events" that represent firings of transitions, but not every set of such events is a process. In Section 4, we define concatenation for arbitrary sets of events. We use it in Section 5 to define finite processes. In Section 6, we define the  $\omega$ -product of finite processes and use it to define infinite processes. We show that this  $\omega$ -product is associative and free. In Section 7, we look at concatenation of infinite processes. Section 8 indicates modifications needed to handle nets with limited capacity of places, and Section 9 relates processes to traces.

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## 2 Associative $\omega$ -products

The set of all natural numbers (positive integers) is in the following denoted by  $\mathbb{N}$ . A sequence of elements of set  $S$  is a mapping  $\mathbf{x} : \mathbb{N} \rightarrow S$ . It is visualized as a linear arrangement of elements  $\mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \dots$ . The set of all sequences of elements of  $S$  is denoted by  $S^{\mathbb{N}}$ . The sequence  $s, \mathbf{x}(1), \mathbf{x}(2), \mathbf{x}(3), \dots$  obtained by adding  $s \in S$  in front of sequence  $\mathbf{x} \in S^{\mathbb{N}}$  is denoted by  $s, \mathbf{x}$ .

A *semigroup* is a pair  $(S, \cdot)$  where  $S$  is a set, and  $\cdot$  is an associative operation on  $S$ , referred to as the *semigroup product*. The product is *left-cancellative* if  $x \cdot y = x \cdot z$  implies  $y = z$  for all  $x, y, z \in S$ .

An  $\omega$ -product on  $(S, \cdot)$  is a surjective mapping  $\pi$  from  $S^{\mathbb{N}}$  to some set  $V$  of *values*. These values may belong to  $S$  or be outside  $S$ .

Given a sequence  $s_1, s_2, s_3, \dots \in S^{\mathbb{N}}$ , it is convenient to write  $\pi(s_1, s_2, s_3, \dots)$  informally as  $s_1 \cdot s_2 \cdot s_3 \cdot \dots$ . In this form, symbol  $\cdot$  denotes the  $\omega$ -product, not an operation on two neighboring factors. Informally speaking, we say that  $\pi$  is "associative" if it satisfies identities of this kind:

$$s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot s_5 \cdot s_6 \dots = (s_1 \cdot s_2 \cdot s_3) \cdot (s_4 \cdot s_5 \cdot s_6) \cdot \dots, \quad (1)$$

$$s_1 \cdot s_2 \cdot s_3 \cdot \dots = (s_1 \cdot \dots \cdot s_n) \cdot (s_{n+1} \cdot s_{n+2} \cdot s_{n+3} \cdot \dots). \quad (2)$$

In (1), we understand the dot within parentheses to mean the semigroup product, and outside parentheses to mean the  $\omega$ -product. In (2), the dot within the first pair of parentheses denotes the semigroup product; within the second pair, it denotes the  $\omega$ -product. The dot in the middle stands for an operation  $\circ : S \times V \rightarrow V$  defined by  $s \circ \pi(\mathbf{x}) = \pi(s, \mathbf{x})$  and called the *mixed product*.

In order to express (1) more precisely, we need the following definition. We say that sequence  $\mathbf{y} = y_1, y_2, y_3, \dots \in S^{\mathbb{N}}$  is a *contraction* of sequence  $\mathbf{x} = x_1, x_2, x_3, \dots \in S^{\mathbb{N}}$ , denoted  $\mathbf{x} \triangleright \mathbf{y}$ , if there exists an ascending sequence of natural numbers  $n_1, n_2, n_3, \dots$  such that

$$y_i = \begin{cases} x_1 \cdot \dots \cdot x_{n_1} & \text{for } i = 1, \\ x_{n_{i-1}+1} \cdot \dots \cdot x_{n_i} & \text{for } i > 1. \end{cases} \quad (3)$$

For example, the sequence  $s_1, (s_2 \cdot s_3), s_4, (s_5 \cdot s_6), \dots$  is a contraction of  $s_1, s_2, s_3, \dots$ . If the semigroup product is left-cancellative, (3) is equivalent to

$$y_1 \cdot \dots \cdot y_i = x_1 \cdot \dots \cdot x_{n_i} \quad \text{for } i \geq 1. \quad (4)$$

Formally, we say that  $\omega$ -product  $\pi : S^{\mathbb{N}} \rightarrow V$  is *associative* if it has these three properties:

$$\mathbf{x} \triangleright \mathbf{y} \Rightarrow \pi(\mathbf{x}) = \pi(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}; \quad (5)$$

$$\pi(\mathbf{x}) = \pi(\mathbf{y}) \Rightarrow \pi(s, \mathbf{x}) = \pi(s, \mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in S^{\mathbb{N}} \text{ and } s \in S; \quad (6)$$

$$\pi(s, \mathbf{x}) = s \cdot \pi(\mathbf{x}) \quad \text{for all } s \in S \text{ and } \mathbf{x} \in S^{\mathbb{N}} \text{ such that } \pi(\mathbf{x}) \in S. \quad (7)$$

Property (5) states validity of all equations of the form (1); property (6) ensures that mixed product is uniquely defined, and (7) ensures that (2) is not ambiguous.

According to (5), we must have  $\pi(\mathbf{x}) = \pi(\mathbf{y})$  whenever  $\mathbf{x}$  and  $\mathbf{y}$  can be transformed into each other in a finite number of steps using contraction. Such sequences  $\mathbf{x}, \mathbf{y}$  are here called *similar*, written  $\mathbf{x} \sim \mathbf{y}$ . More precisely,  $\mathbf{x} \sim \mathbf{y}$  means that there exist sequences  $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_k$  such that  $\mathbf{x} = \mathbf{z}_1$ ,  $\mathbf{y} = \mathbf{z}_k$ , and either  $\mathbf{z}_i \triangleright \mathbf{z}_{i+1}$  or  $\mathbf{z}_{i+1} \triangleright \mathbf{z}_i$  for  $1 \leq i < k$ . In other words,  $\sim$  is the reflexive, symmetric, and transitive closure of  $\triangleright$ ; it is an equivalence relation on  $S^{\mathbb{N}}$ .

Property (5) is identical to  $\mathbf{x} \sim \mathbf{y} \Rightarrow \pi(\mathbf{x}) = \pi(\mathbf{y})$  for all  $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$ . An  $\omega$ -product  $\pi$  such that  $\mathbf{x} \sim \mathbf{y} \Leftrightarrow \pi(\mathbf{x}) = \pi(\mathbf{y})$  and  $\pi(\mathbf{x}) \notin S$  for all  $\mathbf{x}, \mathbf{y} \in S^{\mathbb{N}}$  satisfies all of (5)–(7) and is called a *free associative  $\omega$ -product* on  $(S, \cdot)$ . All free associative  $\omega$ -products on a given semigroup are identical up to the choice of values, in the sense that for any two such products,  $\pi$  and  $\pi'$ , there exists a bijection  $\varphi$  between their sets of values satisfying  $\pi'(\mathbf{x}) = \varphi(\pi(\mathbf{x}))$  for all  $\mathbf{x} \in S^{\mathbb{N}}$ .

### 3 Introduction to processes

A "process" defined in [1–5] is best visualized as a graph like this:

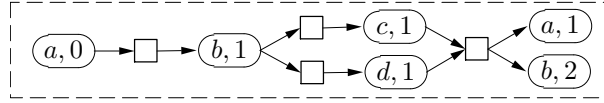


Fig. 1.

Its vertices are "occurrences" and "events". The occurrences are pairs of the form  $\langle x, n \rangle$ . The events are shown as boxes. The graph describes some activity in a Petri net with unlimited place capacity and unlimited initial marking. An occurrence  $\langle x, n \rangle$  represents place  $x$  in the net after it received a token  $n$  times since the start of the activity. An event represents a transition that removes tokens from places identified by incoming arrows and adds tokens to places identified by outgoing arrows. The order of events obtaining tokens from the same occurrence is undefined.

An alternative way to represent the same process is to enumerate its events. An event is identified only by the set of its "inputs" and "outputs". For example, the leftmost event in Figure 1 is only identified as "the event having  $\langle a, 0 \rangle$  as input and  $\langle b, 1 \rangle$  as output". Figure 2 shows all events from Figure 1 identified in this way.

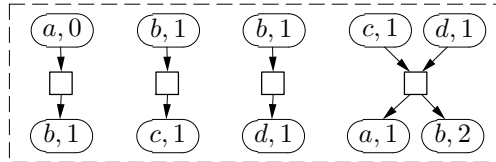
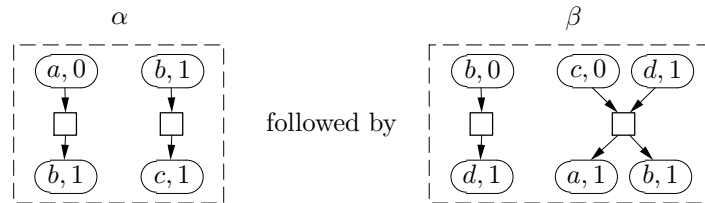


Fig. 2.

As one can easily see, the set of events in Figure 2 contains complete information about the graph of Figure 1: all events, arrows, and occurrences are there. A specific occurrence, such as  $\langle b, 1 \rangle$ , may appear several times, but it is still one occurrence. This method of representing a process is much less intuitive than the graph of Figure 1, but gives a very simple way to express concatenation of processes.

The concatenation  $\alpha \cdot \beta$  of processes  $\alpha$  and  $\beta$  represents, as usual, " $\alpha$  followed by  $\beta$ ". With processes seen as unordered sets of events (such as in Figure 2), this "followed by" is represented by incrementing numbers in the occurrences belonging to  $\beta$ . Indeed, if  $x$  received a token  $n$  times in  $\alpha$ , and  $\beta$  follows  $\alpha$ , an occurrence  $\langle x, m \rangle$  from  $\beta$  represents now  $x$  that received a token  $m + n$  times. This operation of counting up the occurrence numbers in  $\beta$  by the highest occurrence number in  $\alpha$  is called the " $\alpha$ -shift of  $\beta$ " and is denoted by  $\beta^\alpha$ . The concatenation  $\alpha \cdot \beta$  is just the union of  $\alpha$  and  $\beta^\alpha$ , as illustrated in Figure 3.



is defined as  $\alpha \cup \beta^\alpha$  :

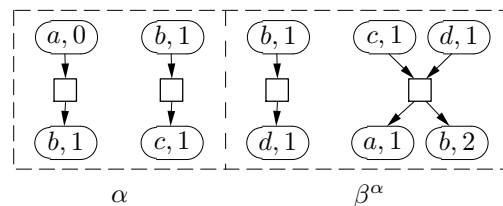


Fig. 3.

(Notice that the highest occurrence numbers of  $a$ ,  $b$ , and  $c$  in  $\alpha$  are, respectively, 0, 1, and 1. As  $d$  does not appear in  $\alpha$ , its occurrence in  $\beta$  need not be adjusted.) Figure 4 shows the same operation with processes represented as graphs. One can see that the two processes have been joined at the places indicated by solid lines.

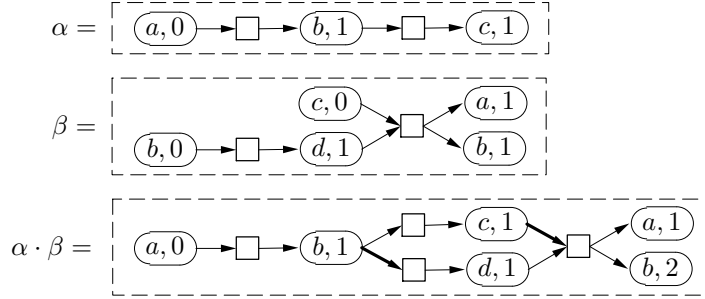


Fig. 4.

The main idea in [1–5] is that processes are constructed by concatenation of "atomic processes". Each atomic process consists of a single event. Its input occurrences are numbered 0 and output occurrences are numbered 1: the input places did not receive a token yet, and each output place received a token once. As an example, the process of Figure 1 is obtained from the four atomic processes in Figure 5, concatenated in the order shown.

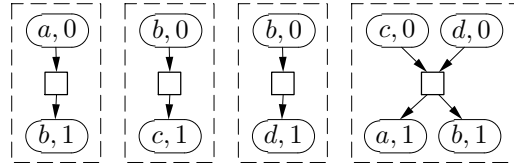


Fig. 5.

## 4 Occurrences, events, concatenation

We proceed now to formal definitions. Let  $\mathbb{X}$  be a given nonempty set of *places* that will remain fixed for the rest of the discussion. An *occurrence* of  $x \in \mathbb{X}$  is a pair  $\langle x, n \rangle$  where  $n$  is a non-negative integer called the *occurrence number*. An occurrence  $\langle x, n \rangle$  with  $n = 0$  is called a *zero-occurrence*; otherwise it is a *non-zero occurrence*.

An *event*  $u$  is an ordered pair  $\langle \bullet u, u^\bullet \rangle$  where  $\bullet u \subseteq \mathbb{X} \times (\mathbb{N} \cup \{0\})$  and  $u^\bullet \subseteq \mathbb{X} \times \mathbb{N}$  are finite nonempty sets. These sets are, respectively, the inputs and outputs of  $u$ . The set of all events is denoted by  $\mathbf{E}$ . Its subsets are in the following denoted by small Greek letters.

Let  $\alpha \subseteq \mathbf{E}$  be any set of events. The set of all occurrences appearing in  $\alpha$  is called the *carrier* of  $\alpha$  and is denoted by  $\text{car}(\alpha)$ :

$$\text{car}(\alpha) = \bigcup_{u \in \alpha} (\bullet u \cup u^\bullet). \quad (8)$$

It is the set of all occurrence vertices of the graph such as in Figure 1. We say that  $\langle x, n \rangle$  belongs to  $\alpha$ , or that  $\alpha$  contains  $\langle x, n \rangle$ , to mean that  $\langle x, n \rangle \in \text{car}(\alpha)$ . We say that place  $x$  occurs in  $\alpha$  to mean that  $\text{car}(\alpha)$  contains an occurrence of  $x$ . The set of all places occurring in  $\alpha$  is denoted by  $\text{pl}(\alpha)$ :

$$\text{pl}(\alpha) = \{x \in \mathbb{X} \mid \langle x, n \rangle \in \text{car}(\alpha) \text{ for some } n\}. \quad (9)$$

For a finite  $\alpha \subseteq \mathbf{E}$  and place  $x \in \mathbb{X}$ , we define  $\Psi(\alpha, x)$  to be the highest number in the set  $\{n \mid \langle x, n \rangle \in \text{car}(\alpha)\} \cup \{0\}$ . We define the  $\alpha$ -*shift* of an event  $u \in \mathbf{E}$  to be the event  $u^\alpha$  such that

$$\begin{aligned} \bullet(u^\alpha) &= \{\langle x, n + \Psi(\alpha, x) \rangle \mid \langle x, n \rangle \in \bullet u\}, \\ (u^\alpha)^\bullet &= \{\langle x, n + \Psi(\alpha, x) \rangle \mid \langle x, n \rangle \in u^\bullet\}. \end{aligned} \quad (10)$$

We extend this to a set  $\beta \subset \mathbf{E}$  by defining

$$\beta^\alpha = \{u^\alpha \mid u \in \beta\}. \quad (11)$$

Finally, we define the *concatenation* of finite  $\alpha \subset \mathbf{E}$  and any  $\beta \subset \mathbf{E}$  as

$$\alpha \cdot \beta = \alpha \cup \beta^\alpha. \quad (12)$$

The processes described in Section 3 are subsets of  $\mathbf{E}$ , but not every such subset is a process. For example, the set of events obtained from that in Figure 2 by changing  $\langle b, 2 \rangle$  to  $\langle b, 3 \rangle$  is not a process. However, the concatenation just defined applies generally to all sets of events, with the only restriction that  $\alpha$  be finite. We shall need a number of facts about this operation.

From (11) follows immediately that  $\alpha$ -shift distributes over unions, that is, for any finite  $\alpha \subset \mathbf{E}$  and any family  $\{\beta_i \subset \mathbf{E} \mid i \in I\}$  holds

$$\left(\bigcup_{i \in I} \beta_i\right)^\alpha = \bigcup_{i \in I} \beta_i^\alpha. \quad (13)$$

**Proposition 1.** *Concatenation is associative, that is,  $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$  holds for any finite  $\alpha, \beta \subset \mathbf{E}$  and any  $\gamma \subset \mathbf{E}$ .*

*Proof.* Let  $\alpha, \beta$  and  $\gamma$  be as stated. From (12) and (13) follows:

$$\begin{aligned} \alpha \cdot (\beta \cdot \gamma) &= \alpha \cup (\beta \cup \gamma^\beta)^\alpha = \alpha \cup \beta^\alpha \cup (\gamma^\beta)^\alpha, \\ (\alpha \cdot \beta) \cdot \gamma &= (\alpha \cdot \beta) \cup \gamma^{\alpha \cdot \beta} = \alpha \cup \beta^\alpha \cup \gamma^{\alpha \cdot \beta}. \end{aligned}$$

It remains to verify that  $(\gamma^\beta)^\alpha = \gamma^{\alpha \cdot \beta}$ . One can easily see that  $\Psi(\alpha \cdot \beta, x) = \Psi(\alpha, x) + \Psi(\beta, x)$  for any  $x \in \mathbb{X}$ . Using this, one can verify that each of the two operations on  $\gamma$  replaces each occurrence  $\langle x, n \rangle$  in  $\gamma$  by  $\langle x, n + \Psi(\alpha, x) + \Psi(\beta, x) \rangle$ .  $\square$

**Proposition 2.** *The sets  $\alpha$  and  $\beta^\alpha$  are disjoint, for any finite  $\alpha \subset \mathbf{E}$  and any  $\beta \subset \mathbf{E}$ .*

*Proof.* Suppose an event  $u$  belongs to both  $\alpha$  and  $\beta^\alpha$ . According to (11), if  $u \in \beta^\alpha$ ,  $\beta$  must contain an event  $v$  such that  $u = v^\alpha$ . Take any  $\langle x, n \rangle \in v^\bullet$ . By definition of event, we have  $n > 0$ . By  $u = v^\alpha$ ,  $u^\bullet$  contains the occurrence  $\langle x, n + \Psi(\alpha, x) \rangle$ . But  $u$  is also in  $\alpha$ , so  $\langle x, n + \Psi(\alpha, x) \rangle \in \text{car}(\alpha)$ ; by definition of  $\Psi(\alpha, x)$  we have  $\Psi(\alpha, x) \geq n + \Psi(\alpha, x)$ , which contradicts  $n > 0$ .  $\square$

**Proposition 3.** *Concatenation is left-cancellative, that is,  $\alpha \cdot \beta = \alpha \cdot \gamma \Rightarrow \beta = \gamma$  for any finite  $\alpha \subset \mathbf{E}$  and any  $\beta, \gamma \subset \mathbf{E}$ .*

*Proof.* Consider any  $\alpha, \beta, \gamma$  such that  $\alpha \cdot \beta = \alpha \cdot \gamma$ . According to (12), that means  $\alpha \cup \beta^\alpha = \alpha \cup \gamma^\alpha$ . By Proposition 2,  $\alpha \cap \beta^\alpha = \alpha \cap \gamma^\alpha = \emptyset$ , so  $\beta^\alpha = \gamma^\alpha$ . One can easily see that  $\alpha$ -shift is reversible, that is, for given  $\alpha$ , the set  $\beta$  is uniquely defined by  $\beta^\alpha$ ; thus  $\beta^\alpha = \gamma^\alpha$  implies  $\beta = \gamma$ .  $\square$

## 5 Finite processes

An *atomic process* is a singleton set  $\{u\} \subset \mathbf{E}$  where  $u$  is an event such that:

- (1)  $n = 0$  for each  $\langle x, n \rangle \in \bullet u$ .
- (2)  $n = 1$  for each  $\langle x, n \rangle \in u^\bullet$ .
- (3) All places occurring in  $\bullet u$  and  $u^\bullet$  are distinct.

A *process* is any concatenation of  $n \geq 1$  atomic processes. (By convention,  $n = 1$  means a single atomic process.) Because of associativity, the concatenation of atomic processes  $\sigma_1, \sigma_2, \dots, \sigma_n$  can be written unambiguously as  $\sigma_1 \cdot \sigma_2 \cdot \dots \cdot \sigma_n$ .

The processes just defined are finite sets of events, and we refer to them as *finite processes*. The set of all finite processes is in the following denoted by  $\mathbf{P}_f$ . Because of associativity, the concatenation of finite processes always results in a finite process, so  $\cdot$  is an operation in  $\mathbf{P}_f$  and  $(\mathbf{P}_f, \cdot)$  is a semigroup.

## 6 Infinite processes

In [1, 2, 4, 5], concatenation is extended to infinite sequences of processes. Let  $\mathbf{a} = \alpha_1, \alpha_2, \alpha_3, \dots$  be a sequence of finite processes. The concatenation of  $\mathbf{a}$  is defined as:

$$\pi(\mathbf{a}) = \bigcup_{i \geq 1} \alpha_1 \cdot \dots \cdot \alpha_i. \quad (14)$$

The result is an infinite set of events and is called an *infinite process*. The set of all infinite processes is in the following denoted by  $\mathbf{P}_\omega$ :

$$\mathbf{P}_\omega = \{\pi(\mathbf{a}) \mid \mathbf{a} \in \mathbf{P}_f^{\mathbb{N}}\}.$$

The mapping  $\pi : \mathbf{P}_f^{\mathbb{N}} \rightarrow \mathbf{P}_\omega$  is, by our definition from Section 2, an  $\omega$ -product on the semigroup  $(\mathbf{P}_f, \cdot)$ . Using (12), one can easily verify that

$$\pi(\mathbf{a}) = \alpha_1 \cup \bigcup_{i > 1} \alpha_i^{\alpha_1 \cdot \dots \cdot \alpha_{i-1}}. \quad (15)$$

**Proposition 4.** *The  $\omega$ -product  $\pi$  satisfies (5).*

*Proof.* Consider any  $\mathbf{a} = \alpha_1, \alpha_2, \alpha_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$  and  $\mathbf{b} = \beta_1, \beta_2, \beta_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$ , such that  $\mathbf{a} \triangleright \mathbf{b}$ . We recall that concatenation is left-cancellative (Proposition 3). According to (4), there exists an ascending sequence of integers  $n_1, n_2, n_3, \dots$  such that  $\beta_1 \cdot \dots \cdot \beta_i = \alpha_1 \cdot \dots \cdot \alpha_{n_i}$  for all  $i \geq 1$ . From (12) follows that  $\alpha_1 \cdot \dots \cdot \alpha_i \subseteq \alpha_1 \cdot \dots \cdot \alpha_j$  whenever  $i < j$ , so the union in (14) will not change if we remove from it all terms  $\alpha_1 \cdot \dots \cdot \alpha_j$  where  $j$  does not appear among  $n_1, n_2, n_3, \dots$ . This gives:

$$\pi(\mathbf{a}) = \bigcup_{i \geq 1} \alpha_1 \cdot \dots \cdot \alpha_i = \bigcup_{i \geq 1} \alpha_1 \cdot \dots \cdot \alpha_{n_i} = \bigcup_{i \geq 1} \beta_1 \cdot \dots \cdot \beta_i = \pi(\mathbf{b}).$$

□

**Proposition 5.** *The mixed product induced by  $\pi$  is identical to the concatenation defined by (12).*

*Proof.* As stated in Section 2, mixed product is an operation  $\circ : \mathbf{P}_f \times \mathbf{P}_\omega \rightarrow \mathbf{P}_\omega$  defined by  $\beta \circ \pi(\mathbf{a}) = \pi(\beta, \mathbf{a})$  for  $\mathbf{a} = \alpha_1, \alpha_2, \alpha_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$  and  $\beta \in \mathbf{P}_f$ . Consider any such  $\mathbf{a}$  and  $\beta$ . Applying (13), we obtain:

$$\begin{aligned} \beta \circ \pi(\mathbf{a}) &= \pi(\beta, \mathbf{a}) = \bigcup_{i \geq 1} (\beta \cdot \alpha_1 \cdot \dots \cdot \alpha_i) = \bigcup_{i \geq 1} (\beta \cup (\alpha_1 \cdot \dots \cdot \alpha_i)^\beta) \\ &= \beta \cup \bigcup_{i \geq 1} (\alpha_1 \cdot \dots \cdot \alpha_i)^\beta = \beta \cup \left( \bigcup_{i \geq 1} \alpha_1 \cdot \dots \cdot \alpha_i \right)^\beta = \beta \cdot \pi(\mathbf{a}). \end{aligned}$$

□

**Proposition 6.**  *$\pi$  is an associative  $\omega$ -product on  $(\mathbf{P}_f, \cdot)$ .*

*Proof.* We recall from Section 2 that  $\pi$  is called associative if it has all of the properties (5)–(7). Property (5) is stated by Proposition 4, and (6) follows immediately from Proposition 5. Property (7) follows from  $\mathbf{P}_f \cap \mathbf{P}_\omega = \emptyset$ . □

Because of (5), each infinite process can be represented as infinite concatenation of atomic processes. Each finite process  $\alpha_i$  in  $\mathbf{a} = \alpha_1, \alpha_2, \alpha_3, \dots$  is a concatenation of some atomic processes. Let  $\mathbf{b}$  be the sequence obtained from  $\mathbf{a}$  by replacing each  $\alpha_i$  by sequence of these atomic processes. Obviously,  $\mathbf{a}$  is a contraction of  $\mathbf{b}$ , so  $\pi(\mathbf{a}) = \pi(\mathbf{b})$ . The sequence  $\mathbf{b}$  is in the following called a *generating sequence* of  $\pi(\mathbf{a})$ . The process  $\pi(\mathbf{a})$  can, in general, have many generating sequences. It turns out that these sequences are all similar, in the sense defined in Section 2. To demonstrate this, we need a couple of auxiliary results.

We shall say that processes  $\alpha \in \mathbf{P}_f$  and  $\beta \in \mathbf{P}_f$  are *independent* if either  $\text{pl}(\alpha) \cap \text{pl}(\beta) = \emptyset$ , or all occurrences in  $\alpha$  and  $\beta$  of places occurring in both are zero-occurrences. Thus, if  $\alpha$  and  $\beta$  are independent, we have  $\Psi(\alpha, x) = 0$  for each  $x \in \text{pl}(\beta)$  and  $\Psi(\beta, x) = 0$  for each  $x \in \text{pl}(\alpha)$ , from which follows  $\alpha^\beta = \alpha$ ,  $\beta^\alpha = \beta$  and  $\alpha \cdot \beta = \alpha \cup \beta = \beta \cdot \alpha$ .

**Lemma 1.** Let  $\mathbf{s} = \sigma_1, \sigma_2, \sigma_3, \dots$  and  $\mathbf{t} = \tau_1, \tau_2, \tau_3, \dots$  be two sequences of atomic processes such that  $\pi(\mathbf{s}) = \pi(\mathbf{t})$ . There exists  $i \geq 1$  such that  $\tau_i = \sigma_1$  and  $\tau_1 \cdot \dots \cdot \tau_i = \tau_i \cdot \tau_1 \cdot \dots \cdot \tau_{i-1}$ .

*Proof.* The Lemma is trivially true if  $\sigma_1 = \tau_1$ . Assume  $\sigma_1 \neq \tau_1$ . According to (15),  $\pi(\mathbf{s})$  contains unshifted event  $u$  from  $\sigma_1$ . But  $\pi(\mathbf{s}) = \pi(\mathbf{t})$ , so  $u$  appears also in  $\pi(\mathbf{t})$ . We have thus  $u \in \tau_i^\alpha$  for some  $i > 1$ , where  $\alpha = \tau_1 \cdot \dots \cdot \tau_{i-1}$ . As  $u$  is an unshifted event from an atomic process, we must have  $\Psi(\alpha, x) = 0$  for each  $x \in \text{pl}(\tau_i)$ . That means, either  $x$  does not occur in  $\alpha$  or all occurrences of  $x$  in  $\alpha$  are zero-occurrences.

If none of  $x \in \text{pl}(\tau_i)$  occurs in  $\alpha$ ,  $\alpha$  and  $\tau_i$  are independent, and  $\alpha \cdot \tau_i = \tau_i \cdot \alpha$ .

Suppose some  $x \in \text{pl}(\tau_i)$  occurs in  $\alpha$ . As found above, that occurrence of  $x$  must be a zero-occurrence. Let  $v$  be the event containing that occurrence. Because  $\pi(\mathbf{s}) = \pi(\mathbf{t})$ ,  $v$  belongs also to  $\pi(\mathbf{s})$ . That means  $v$  is the event from  $\sigma_j$  for some  $j \geq 1$ , shifted by  $\beta = \sigma_1 \cdot \dots \cdot \sigma_{j-1}$ . But the occurrence  $\langle x, 0 \rangle$  in  $v$  can only be the result of shift by 0, meaning  $\Psi(\beta, x) = 0$ . Hence, any occurrence of  $x$  in  $\beta$  must be a zero-occurrence. This applies, in particular, to the occurrence of  $x$  in  $\sigma_1 = \tau_i$ . It follows that any occurrence of  $x$  in both  $\alpha$  and  $\tau_i$  is a zero-occurrence. Thus,  $\alpha$  and  $\tau_i$  are independent, and  $\alpha \cdot \tau_i = \tau_i \cdot \alpha$ . As  $\alpha = \tau_1 \cdot \dots \cdot \tau_{i-1}$ , we have in each case  $\tau_1 \cdot \dots \cdot \tau_i = \tau_i \cdot \tau_1 \cdot \dots \cdot \tau_{i-1}$ .  $\square$

**Lemma 2.** Let  $\mathbf{s} = \sigma_1, \sigma_2, \sigma_3, \dots$  and  $\mathbf{t} = \tau_1, \tau_2, \tau_3, \dots$  be two sequences of atomic processes such that  $\pi(\mathbf{s}) = \pi(\mathbf{t})$ . For each  $n \geq 1$  there exist  $m > n$  and  $\gamma \in \mathbf{P}_f$  such that  $\sigma_1 \cdot \dots \cdot \sigma_n \cdot \gamma = \tau_1 \cdot \dots \cdot \tau_m$ .

*Proof.* Take any  $n \geq 1$ . By a repeated application of Lemma 1, we arrive at a permutation  $\tau'_1 \cdot \dots \cdot \tau'_m$  of  $\tau_1 \cdot \dots \cdot \tau_m$  such that  $\tau'_1 \cdot \dots \cdot \tau'_m = \tau_1 \cdot \dots \cdot \tau_m$  and  $\tau'_i = \sigma_i$  for  $1 \geq i \geq n$ . We can always take  $m > n$ , so we have  $\sigma_1 \cdot \dots \cdot \sigma_n \cdot \tau'_{n+1} \cdot \dots \cdot \tau'_m = \tau_1 \cdot \dots \cdot \tau_m$ . This is the stated result with  $\gamma = \tau'_{n+1} \cdot \dots \cdot \tau'_m$ .  $\square$

**Proposition 7.** For any sequences  $\mathbf{s}, \mathbf{t}$  of atomic processes,  $\pi(\mathbf{s}) = \pi(\mathbf{t})$  implies  $\mathbf{s} \sim \mathbf{t}$ .

*Proof.* Let  $\mathbf{s} = \sigma_1, \sigma_2, \sigma_3, \dots$  and  $\mathbf{t} = \tau_1, \tau_2, \tau_3, \dots$  where  $\sigma_i$  and  $\tau_i$  are atomic for  $i \geq 1$ . Take any  $n_1 \geq 1$ . According to Lemma 2, there exist:

- $m_1 > n_1$  and  $\gamma_1$  such that  $\sigma_1 \cdot \dots \cdot \sigma_{n_1} \cdot \gamma_1 = \tau_1 \cdot \dots \cdot \tau_{m_1}$ ,
- $n_2 > m_1$  and  $\delta_1$  such that  $\tau_1 \cdot \dots \cdot \tau_{m_1} \cdot \delta_1 = \sigma_1 \cdot \dots \cdot \sigma_{n_2}$ ,
- $m_2 > n_2$  and  $\gamma_2$  such that  $\sigma_1 \cdot \dots \cdot \sigma_{n_2} \cdot \gamma_2 = \tau_1 \cdot \dots \cdot \tau_{m_2}$ ,

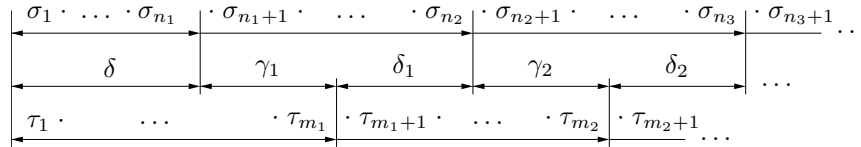
and so on. This gives two ascending sequences,  $n_1, n_2, n_3, \dots$  and  $m_1, m_2, m_3, \dots$ , of natural numbers and two sequences,  $\gamma_1, \gamma_2, \gamma_3, \dots$  and  $\delta_1, \delta_2, \delta_3, \dots$ , of finite processes, such that

$$\begin{aligned} \sigma_1 \cdot \dots \cdot \sigma_{n_i} \cdot \gamma_i &= \tau_1 \cdot \dots \cdot \tau_{m_i}, \\ \tau_1 \cdot \dots \cdot \tau_{m_i} \cdot \delta_i &= \sigma_1 \cdot \dots \cdot \sigma_{n_{i+1}}, \end{aligned}$$

for  $i \geq 1$ . Denoting  $\delta = \sigma_1 \cdot \dots \cdot \sigma_{n_1}$  we obtain, by induction:

$$\begin{aligned} \sigma_1 \cdot \dots \cdot \sigma_{n_i} &= \delta \cdot \gamma_1 \cdot \delta_1 \cdot \dots \cdot \gamma_{i-1} \cdot \delta_{i-1}, \\ \tau_1 \cdot \dots \cdot \tau_{m_i} &= \delta \cdot \gamma_1 \cdot \delta_1 \cdot \gamma_2 \cdot \dots \cdot \delta_{i-1} \cdot \gamma_i, \end{aligned}$$

for  $i \geq 1$ . This can be illustrated as follows:



**Fig. 6.**

Define:

$$\begin{aligned} \mathbf{z} &= \delta, \gamma_1, \delta_1, \gamma_2, \delta_2, \dots, \\ \mathbf{p} &= \delta, (\gamma_1 \cdot \delta_1), (\gamma_2, \delta_2), \dots, \\ \mathbf{q} &= (\delta \cdot \gamma_1), (\delta_1 \cdot \gamma_2), (\delta_2 \cdot \delta_3), \dots \end{aligned}$$

One can easily see that  $\mathbf{z} \triangleright \mathbf{p}$  and  $\mathbf{z} \triangleright \mathbf{q}$ . Recalling that concatenation is left-cancellative and using (4), we obtain  $\mathbf{s} \triangleright \mathbf{p}$  and  $\mathbf{t} \triangleright \mathbf{q}$ . We have thus  $\mathbf{s} \triangleright \mathbf{p} \triangleleft \mathbf{z} \triangleright \mathbf{q} \triangleleft \mathbf{t}$  and  $\mathbf{s} \sim \mathbf{t}$ .  $\square$

**Proposition 8.**  $\pi$  is a free associative  $\omega$ -product on  $(\mathbf{P}_f, \cdot)$ .

*Proof.* We recall from Section 2 that  $\pi$  is free if  $\mathbf{a} \sim \mathbf{b} \Leftrightarrow \pi(\mathbf{a}) = \pi(\mathbf{b})$  and  $\pi(\mathbf{a}) \notin \mathbf{P}_f$  for all  $\mathbf{a}, \mathbf{b} \in \mathbf{P}_f^{\mathbb{N}}$ . The first property follows from Propositions 4 and 7. The second follows from  $\mathbf{P}_f \cap \mathbf{P}_\omega = \emptyset$ .  $\square$

## 7 Concatenation of infinite processes

In the concatenation  $\alpha \cdot \beta$  considered so far,  $\alpha$  was always a finite process. Indeed, an infinite process with something added behind would, in general, be a "transfinite" process – a concept considered in [2] to be of a "lower intuitive appeal". However, in some cases  $\alpha \cdot \beta$  with infinite  $\alpha$  makes sense without becoming transfinite, as in the following example:

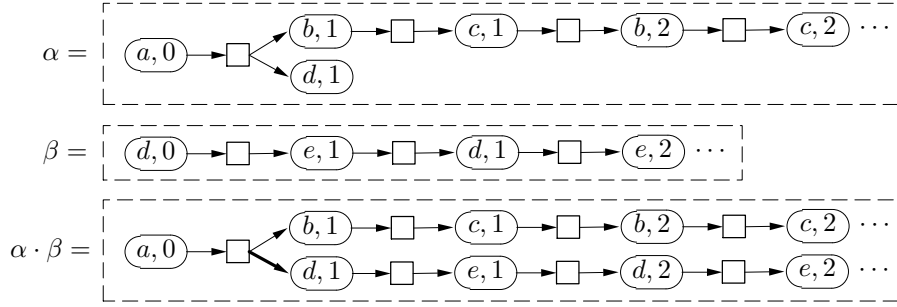


Fig. 7.

The reason why this makes sense is that the two processes are largely independent. An interesting result from [2] is that if two infinite processes are sufficiently independent, their concatenation is also a process: an infinite concatenation of their generating sequences suitably interleaved. In the following, we attempt to simplify the proof of that result by exploiting associativity.

We note first that if  $\text{car}(\alpha)$  contains at most finitely many non-zero occurrences of a place  $x \in \mathbb{X}$ , the definition of  $\Psi(\alpha, x)$  as the highest  $n$  in  $\{n \mid \langle x, n \rangle \in \text{car}(\alpha)\} \cup \{0\}$  makes perfect sense, even if  $\alpha$  is infinite. If this is true for all  $x \in \text{pl}(\beta)$ , our definitions (10), (11) and (12) of  $\beta^\alpha$  and  $\alpha \cdot \beta$  are valid, and we continue to apply them in such a case. They describe, in particular, the operation illustrated in Figure 7.

If  $\text{car}(\alpha)$  contains infinitely many non-zero occurrences of  $x$ , there is no such highest number. If this is the case for any  $x \in \text{pl}(\beta)$ , the definitions of  $\beta^\alpha$  and  $\alpha \cdot \beta$  are no longer valid. We indicate this by writing  $\alpha \cdot \beta = \Omega$  where  $\Omega$  is a new object meaning "undefined". In [1, 2, 4, 5], one continues by extending concatenation to a semigroup containing  $\Omega$ , but this is not needed here.

The notion of independence extends unchanged to infinite processes, and we have  $\alpha \cdot \beta = \beta \cdot \alpha = \alpha \cup \beta \neq \Omega$  for any independent  $\alpha, \beta \in \mathbf{P}_f \cup \mathbf{P}_\omega$ .

**Proposition 9.** Let  $\mathbf{a} = \alpha_1, \alpha_2, \alpha_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$  and  $\mathbf{b} = \beta_1, \beta_2, \beta_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$  be such that, for each  $i \geq 1$ ,  $\beta_i$  is independent of all  $\alpha_j$  with  $j > i$ . Then  $\pi(\mathbf{a}) \cdot \pi(\mathbf{b}) = \pi(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \dots) \neq \Omega$ .

*Proof.* Let  $\mathbf{a}$  and  $\mathbf{b}$  be as stated. Consider any  $n \geq 1$ . One can easily see that  $\beta_1 \cdot \dots \cdot \beta_n$  and  $\pi(\alpha_{n+1}, \alpha_{n+2}, \dots)$  are independent. That means  $\Psi(\pi(\alpha_{n+1}, \alpha_{n+2}, \dots), x) = 0$  for each  $x \in \text{pl}(\beta_1 \cdot \dots \cdot \beta_n)$ . But  $\pi(\mathbf{a}) = (\alpha_1 \cdot \dots \cdot \alpha_n) \cdot \pi(\alpha_{n+1}, \alpha_{n+2}, \dots)$ , so we have, for each  $x \in \text{pl}(\beta_1 \cdot \dots \cdot \beta_n)$ :

$$\Psi(\pi(\mathbf{a}), x) = \Psi(\alpha_1 \cdot \dots \cdot \alpha_n, x) + \Psi(\pi(\alpha_{n+1}, \alpha_{n+2}, \dots), x) = \Psi(\alpha_1 \cdot \dots \cdot \alpha_n, x).$$

From this follows, in particular, that  $(\beta_1 \cdot \dots \cdot \beta_n)^{\pi(\mathbf{a})} = (\beta_1 \cdot \dots \cdot \beta_n)^{\alpha_1 \cdot \dots \cdot \alpha_n}$ .

Clearly, each  $x \in \text{pl}(\mathbf{b})$  belongs to  $\text{pl}(\beta_1 \cdot \dots \cdot \beta_n)$  for some  $n \geq 1$ ; that means  $\Psi(\pi(\mathbf{a}), x)$  is defined for each such  $x$ , so  $\pi(\mathbf{a}) \cdot \pi(\mathbf{b}) \neq \Omega$ . From independence of  $\beta_1 \cdot \dots \cdot \beta_n$  and  $\pi(\alpha_{n+1}, \alpha_{n+2}, \dots)$  follows:

$$\begin{aligned} \pi(\mathbf{a}) \cdot \pi(\mathbf{b}) &= (\alpha_1 \cdot \dots \cdot \alpha_n) \cdot \pi(\alpha_{n+1}, \alpha_{n+2}, \dots) \cdot (\beta_1 \cdot \dots \cdot \beta_n) \cdot \pi(\beta_{n+1}, \beta_{n+2}, \dots) \\ &= (\alpha_1 \cdot \dots \cdot \alpha_n) \cdot (\beta_1 \cdot \dots \cdot \beta_n) \cdot \pi(\alpha_{n+1}, \alpha_{n+2}, \dots) \cdot \pi(\beta_{n+1}, \beta_{n+2}, \dots), \end{aligned}$$

showing that  $(\alpha_1 \cdot \dots \cdot \alpha_n) \cdot (\beta_1 \cdot \dots \cdot \beta_n) \subseteq \pi(\mathbf{a}) \cdot \pi(\mathbf{b})$  for all  $n \geq 1$ . Take now any event  $u \in \pi(\mathbf{a}) \cdot \pi(\mathbf{b})$ . It either belongs to  $\pi(\mathbf{a})$  and thus to  $(\alpha_1 \cdot \dots \cdot \alpha_n) \subseteq (\alpha_1 \cdot \dots \cdot \alpha_n) \cdot (\beta_1 \cdot \dots \cdot \beta_n)$  for some  $n \geq 1$ ; or it



belongs to  $\pi(\mathbf{b})^{\pi(\mathbf{a})}$  and thus to  $(\beta_1 \dots \beta_n)^{\pi(\mathbf{a})} = (\beta_1 \dots \beta_n)^{\alpha_1 \dots \alpha_n} \subseteq (\alpha_1 \dots \alpha_n) \cdot (\beta_1 \dots \beta_n)$  for some  $n \geq 1$ . This proves  $\pi(\mathbf{a}) \cdot \pi(\mathbf{b}) = \bigcup_{n \geq 1} (\alpha_1 \dots \alpha_n) \cdot (\beta_1 \dots \beta_n)$ . By assumption about independence,  $(\alpha_1 \dots \alpha_n) \cdot (\beta_1 \dots \beta_n) = \alpha_1 \cdot \beta_1 \dots \alpha_n \cdot \beta_n$ . We have thus

$$\pi(\mathbf{a}) \cdot \pi(\mathbf{b}) = \bigcup_{n \geq 1} (\alpha_1 \cdot \beta_1 \dots \alpha_n \cdot \beta_n) = \pi(\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3, \dots).$$

□

Let us say that  $\mathbf{b} = \beta_1, \beta_2, \beta_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$  is *almost independent* of  $\mathbf{a} = \alpha_1, \alpha_2, \alpha_3, \dots \in \mathbf{P}_f^{\mathbb{N}}$  to mean that each  $\beta_i$  is independent of all but finitely many  $\alpha_j$ .

**Proposition 10.** *If  $\mathbf{b}$  is almost independent on  $\mathbf{a}$ , there exists an ascending sequence of natural numbers  $n_1, n_2, n_3, \dots$  such that*

$$\pi(\mathbf{a}) \cdot \pi(\mathbf{b}) = \pi(\alpha_1, \dots, \alpha_{n_1}, \beta_1, \alpha_{n_1+1}, \dots, \alpha_{n_2}, \beta_2, \alpha_{n_2+1}, \dots, \alpha_{n_3}, \beta_3, \dots) \neq \Omega.$$

*Proof.* If  $\mathbf{b}$  is almost independent of  $\mathbf{a}$ , there exists an ascending sequence of natural numbers  $n_1, n_2, n_3, \dots$  such that for each  $i \geq 1$ ,  $\beta_i$  is independent of all  $\alpha_j$  with  $j > n_i$ . Define  $\mathbf{c} = \gamma_1, \gamma_2, \gamma_3, \dots$  as the following contraction of  $\mathbf{a}$ :

$$\gamma_i = \begin{cases} \alpha_1 \dots \alpha_{n_1} & \text{for } i = 1, \\ \alpha_{n_{i-1}+1} \dots \alpha_{n_i} & \text{for } i > 1; \end{cases}$$

Clearly,  $\beta_i$  is independent of all  $\gamma_j$  with  $j > i$ .

From Proposition 9 follows  $\pi(\mathbf{c}) \cdot \pi(\mathbf{b}) = \pi(\gamma_1, \beta_1, \gamma_2, \beta_2, \gamma_3, \beta_3, \dots) \neq \Omega$ . The stated result follows by Proposition 4. □

## 8 Processes in marked nets

As indicated in Section 3, the processes discussed so far represent behavior of Petri nets with unlimited capacity of places and unlimited initial marking. They are referred to in [5] as "processes in unmarked nets".

The limited capacity and initial marking can be handled in two different ways. In [1,2], concatenation is restricted so that the result is "undefined" if it does not correspond to a possible firing sequence. In [3–5], one defines two functions,  $I_\alpha(x)$  and  $O_\alpha(x)$ , that represent the number of tokens added to, respectively removed from, place  $x$  by process  $\alpha$ . These functions are used to formulate a condition that identifies possible processes.

Each of these methods defines the subset  $\mathbf{M}_f \subseteq \mathbf{P}_f$  of finite processes that are possible in a specific marked net. Denoting the impossible processes by  $\Omega$ , one obtains a semigroup  $(\mathbf{M}_f \cup \{\Omega\}, \circ)$ , where  $\circ$  is an associative operation defined for  $\alpha, \beta \in \mathbf{M}_f$  as  $\alpha \circ \beta = \alpha \cdot \beta$  if  $\alpha \cdot \beta \in \mathbf{M}_f$ ,  $\alpha \circ \beta = \Omega$  if  $\alpha \cdot \beta \notin \mathbf{M}_f$ , and  $\alpha \circ \Omega = \Omega \circ \alpha = \Omega \circ \Omega = \Omega$ .

An infinite process is possible if and only if all its initial portions are possible. This is reflected by defining, for  $\alpha_1, \alpha_2, \alpha_3, \dots \in \mathbf{M}_f^{\mathbb{N}}$ ,  $\pi'(\alpha_1, \alpha_2, \alpha_3, \dots) = \pi(\alpha_1, \alpha_2, \alpha_3, \dots)$  if  $\alpha_1 \circ \dots \circ \alpha_i \neq \Omega$  for all  $i \geq 1$ , and  $\pi'(\alpha_1, \alpha_2, \alpha_3, \dots) = \Omega$  otherwise. The  $\omega$ -product  $\pi'$  on  $(\mathbf{M}_f \cup \{\Omega\}, \circ)$  is associative, but not free.

## 9 Processes and traces

It is interesting to compare the above results with similar results for traces. Traces [9,12] are another way to describe concurrent systems. In [17], the present author considered four different versions of a free associative  $\omega$ -product of traces. Such  $\omega$ -products are natural candidates for the definition of infinite trace. In fact, two of them are identical to the notions of infinite trace suggested in [8,10,11]. All four are complicated constructions based on prefixes.

As shown in [3], the semigroup of finite processes is isomorphic with the semigroup of finite traces. The free associative  $\omega$ -products on these semigroups are thus also isomorphic. In view of this fact, it is surprising that infinite process can be defined in a very simple and intuitive manner: just an infinite union of partial products – a natural limit of the sequence  $\alpha_1 \subseteq \alpha_1 \cdot \alpha_2 \subseteq \alpha_1 \cdot \alpha_2 \cdot \alpha_3 \subseteq \dots$ .

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