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Automata Theory: Infinite Computations

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SEMINAR-REPORT

Adding an Infinite Product to a Semigroup

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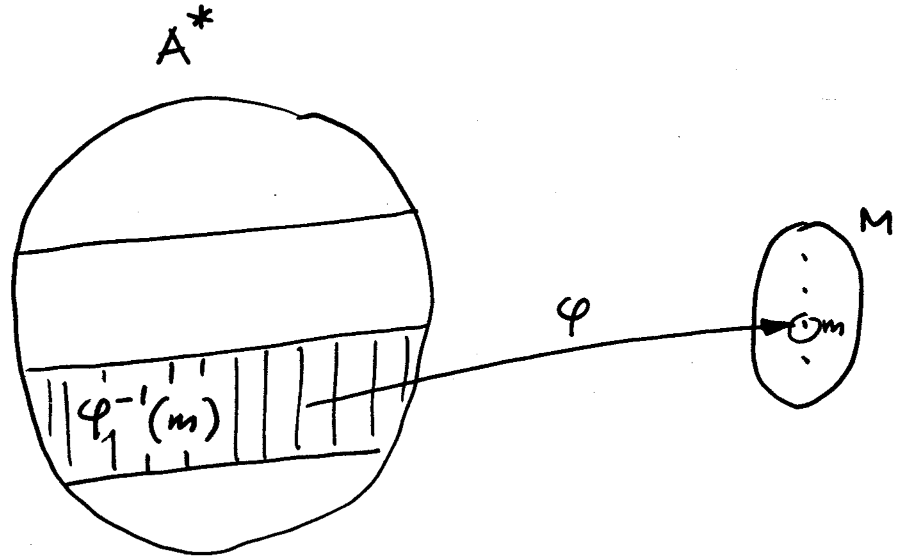
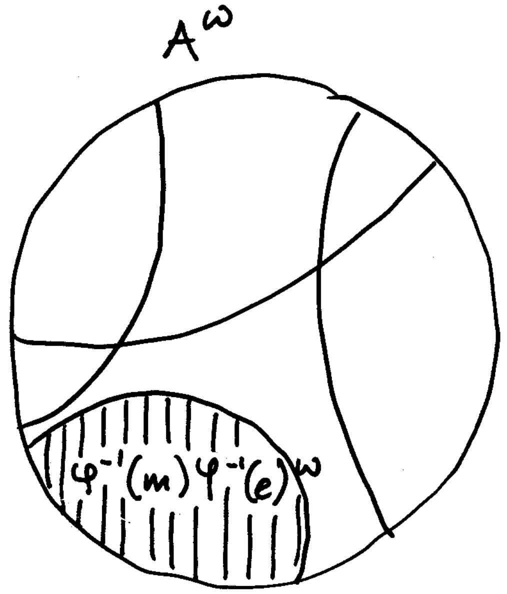
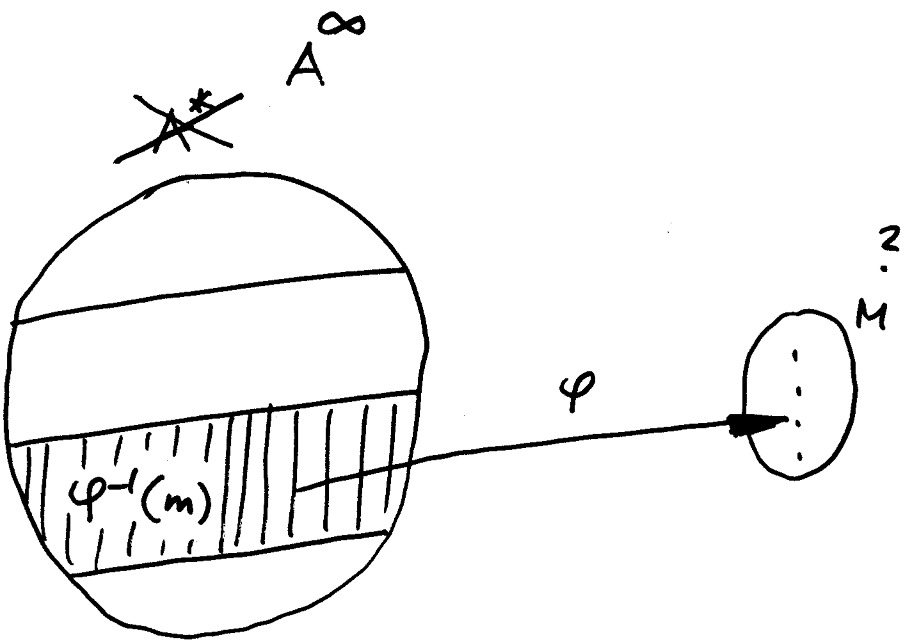
A semigroup $(S,\cdot)$ is extended with an ω-argument operation Π: $S^N \to Q$ where Q is a certain set, possibly disjoint with S. The operation Π is required to satisfy three axioms, expressing generalized associativity compatible with the semigroup operation.

It follows from the axioms and the Ramsey lemma that for a finite S, Π can have only a finite number of distinct values, all reducible to the form se^{ω} where $s,e \in S$ and $e^2=e$.

Homomorphism and a unique free algebra are defined for the class of semigroups extended with the operation Π. The purpose of the construction is to express recognizability of ω-languages in a way analogous to that of finite-word languages (directly in terms of a homomorphism into a finite algebraic structure).


A*
M
φ
m
φ⁻¹(m)


Aω
φ⁻¹(m)φ⁻¹(e)ω



<u>Problem:</u> How to define φ?
Words in A^ω are infinite products of letters.
These products must be mirrored in M.
What is the product of a sequence of elements of M?

<u>Ex.</u> $M = (\{a,b\}, \cdot)$

	a	b
a	a	b
b	b	b

What is $b \cdot a \cdot a \cdot a \cdot a \cdot \ldots\ldots$?

☐ a (has almost only a)

☐ b (any initial finite product gives b)

Given semigroup $(S, \cdot)$

Add ω-product (Q, π)

where:

- Q is any set
- $\pi : Seq(S) \to Q$ a function onto

Example 1

$S = (A^+, \cdot)$ for some alphabet A

$Q = A^\omega$

$\pi(w_1, w_2, w_3 \dots) = w_1 w_2 w_3 \dots$ (concatenation)

($w_i \in A^+$).

Example 2

$S = (A^*, \cdot)$

$Q = A^\omega \cup A^* z$ where $z \notin A$

If $w_1, w_2, w_3, \dots = w_1, w_2, \dots w_n, 1, 1, 1 \dots$

for some n, then

$\pi(w_1, w_2, w_3, \dots) = w_1 w_2 \dots w_n z$

Otherwise

$\pi(w_1, w_2, w_3, \dots) = w_1 w_2 w_3 \dots$

Example 3

Same as Example 2, but

$\pi(w_1, w_2, \dots, w_n, 1, 1, 1, \dots) = w_1 w_2 \dots w_n$

$(Q = A^\omega \cup A^*)$

Example 4

$S = (2^X, \cup)$ for some space X

$Q = 2^X$

$\Pi(X_1, X_2, X_3, \dots) = \bigcup_{i \in N} X_i$

(where $X_i \subseteq X$).

Example 5

$S = (\{a,b\}, \cdot)$ where $\cdot$ is

$\cdot$	a	b
a	a	b
b	b	b

$Q = \{a,b\}$

For $x \in Seq(S)$:

$$\Pi(x) = \begin{cases} a & \text{if } x = a,a,a,\dots \\ b & \text{otherwise} \end{cases}$$

Example 6

$S = (\{0,1\}, \vee)$

$Q = \{A, B\}$

For $x \in Seq(S)$:

$$\Pi(x) = \begin{cases} A & \text{if } x = 0,0,0,\dots \\ B & \text{otherwise} \end{cases}$$

Example 7

$S = (\mathbb{R}_+, +)$ $\qquad$ $\mathbb{R}_+$ - real numbers ≥ 0

$Q = \mathbb{R}_+ \cup \{\infty\}$

For $r = r_1, r_2, r_3, \ldots$ $\qquad$ $(r_i \in \mathbb{R}_+)$:

$$\Pi(r) = \begin{cases} \Sigma r_i & \text{if } r \text{ converges} \\ \infty & \text{otherwise} \end{cases}$$

Example 8

Same as Ex 7, but $\mathbb{R}$ instead of $\mathbb{R}_+$

Example 9

$S = (\{0,1\}, \vee)$

$Q = \{0,1\}$

For $x \in Seq(S)$:

$$\Pi(x) = \begin{cases} 1 & \text{if } x = 0,0,0,\ldots \\ 0 & \text{otherwise} \end{cases}$$

Example 10

$S = (\{0,1\}, \vee)$

$Q = \{A, B\}$

For $x \in Seq(S)$:

$$\Pi(x) = \begin{cases} A & \text{if } x \text{ contains none or inf. many 1's} \\ B & \text{otherwise} \end{cases}$$

- What is wrong with Example 10?

$$\pi(0,0,0,\dots) = A = \pi(1,1,1,\dots)$$
$$\pi(1,0,0,0,\dots) = B \underline{\neq} \pi(1,1,1,1,\dots)$$

$\longrightarrow$ We want $\pi(x) = \pi(y) \Rightarrow \pi(s,x) = \pi(s,y)$

- What is wrong with Example 9?

$$\pi(1,0,0,0,\dots) = 0$$
$$1 \vee \pi(0,0,0,\dots) = 1 \vee 1 = 1 \underline{\underline{\neq}} 0$$

$\longrightarrow$ We want $\pi(x) \in S \Rightarrow s \cdot \pi(x) = \pi(s,x)$

- What is wrong with Example 8?

$$\pi(-1,+1,-1,+1,\dots) = \infty$$
$$\pi((-1+1),(-1+1),\dots) = \pi(0,0,0,\dots) = 0 \underline{\neq} \infty$$

$\longrightarrow$ We want "infinite associativity"

Definition

Let $x \in Seq(S)$
$n \in Seq(N)$ be strictly increasing

$x|n$ (x reduced by n) is a sequence $y \in Seq(S)$

defined by

$$y_i = \begin{cases} x_1 \cdot x_2 \cdot \ldots \cdot x_{n_1} & \text{for } i=1 \\ x_{n_{i-1}+1} \cdot x_{n_{i-1}+2} \cdot \ldots \cdot x_{n_i} & \text{for } i>1 \end{cases}$$

Examples

$S = \{a, b\}$

	a	b
a	a	b
b	b	b

$$(a,b,a,b,a,b \ldots)|(3,6,9,\ldots) = a \cdot b \cdot a,\ b \cdot a \cdot b, \ldots$$
$$= b, b, b, \ldots$$

$$(a,b,a,b,a,b,\ldots)|(2,3,5,6,\ldots) = a \cdot b,\ a,\ b \cdot a,\ b, \ldots$$
$$= b, a, b, b, \ldots$$

$x|n$ is a <u>reduction</u> of x

x is an <u>expansion</u> of $x|n$

(A1) $\pi(x) = \pi(y) \Rightarrow \pi(s,x) = \pi(s,y)$

FOR $x, y \in Seq(S)$, $s \in S$

(A2) $\pi(x) \in S \Rightarrow s \cdot \pi(x) = \pi(s,x)$

FOR $x \in Seq(S)$, $s \in S$

(A3) π IS INFINITELY ASSOCIATIVE

(A3) $\pi(x) = \pi(x|n)$

FOR $x \in Seq(X)$
$n \in Seq(N)$
n STRICTLY ASCENDING

<u>Definition</u>:
Extended semigroup $(S, Q, \cdot, \pi)$
- $(S, \cdot)$ is a semigroup
- (Q, π) is an ω-product over $(S, \cdot)$ satisfying (A1)-(A3)

If (A1) holds,
for each $s \in S$, $q \in Q$ exists unique
$s \bullet q = \pi(s, x)$ where $\pi(x) = q$.

Write informally $\pi(x_1, x_2, x_3 \dots)$ as $x_1 \cdot x_2 \cdot x_3 \cdot \dots$

(A1) permits things like this:

$$x_1 \cdot x_2 \cdot x_3 \cdot \dots = x_1 \bullet (x_2 \bullet (x_3 \cdot x_4 \cdot \dots))$$

(A2) permits things like this:

$$x_1 \cdot x_2 \cdot x_3 \dots = (x_1 \cdot x_2) \bullet (x_3 \cdot x_4 \cdot \dots)$$

(A3) permits things like this:

$$x_1 \cdot x_2 \cdot x_3 \cdot \dots = (x_1 \cdot x_2) \cdot (x_3 \cdot x_4) \cdot \dots$$

Definition

Let $x, y \in Seq(S)$.

$x \sim y$ (x and y are similar)

means: x can be transformed into y by a sequence of reductions and/or expansions.

Properties

- $\sim$ is an equivalence
- $x \sim y \Rightarrow s,x \sim s,y$ for any $s \in S$
- If $\varphi : S \to S'$ is a homomorphism then $x \sim y \Rightarrow \varphi x \sim \varphi y$
- (From Ramsay) If S is finite then for every $x \in Seq(S)$ exists reduction $x | n = s, e, e, e, \dots$ where e is an idempotent of S
- If S is finite then the number of equivalence classes of $\sim$ is finite

Consequences

- (A3) is equivalent to $x \sim y \Rightarrow \pi(x) = \pi(y)$
- In any extended semigroup $(S, Q, \cdot, \pi)$, if S is finite, so is Q.

Definition

Let $(S, Q, \cdot, \Pi)$, $(S', Q', \circ, \Pi')$ be extended semigroups.

$\varphi : (S \cup Q) \to (S' \cup Q')$

is a homomorphism if:

- $\varphi(S) \subseteq S'$
- $\varphi(Q) \subseteq Q'$
- $\varphi(s_1 \cdot s_2) = \varphi(s_1) \circ \varphi(s_2)$ for $s_1, s_2 \in S$
- $\varphi(\Pi(x)) = \Pi'(\varphi(x))$ for $x \in Seq(S)$

Definition

An ω-product (Q, π) over a semigroup $(S, \cdot)$ is <u>free</u> if:

- $S \cap Q = \emptyset$
- $x \sim y \Leftrightarrow \pi(x) = \pi(y)$ for all $x, y \in Seq(S)$

Properties

- A free ω-product satisfies (A1 – A3)
- A free ω-product over a given $(S, \cdot)$ is unique (up to isomorphism)
- Let $(S, Q, \cdot, \pi)$ be an extended semigroup such that (Q, π) is free.
  Let $(S', Q', \circ, \pi')$ be any extended semigroup.
  Then, each homomorphism $\varphi : S \to S'$ has a unique extension to homomorphism $(S \cup Q) \to (S' \cup Q')$.

Should be easy to prove something like:

$L \subseteq A^\infty$ is recognizable

iff

there exists finite extended semigroup $(S, Q, \cdot, \pi)$ and a homomorphism $\varphi: A^\infty \to S \cup Q$ such that L is a union of classes $\varphi^{-1}(s)$ for some $s \in S \cup Q$.