Theoretical Computer Science 43 (1986) 59-79
North-Holland

INFINITE-WORD LANGUAGES AND CONTINUOUS MAPPINGS

Roman R. REDZIEJOWSKI
IBM Nordic Laboratory, S-18109 Lidingö, Sweden

Communicated by M. Nivat
Received March 1985
Revised June 1985

Abstract. Languages with infinite words have often been studied as subsets of a topological space. The topology used for this purpose was always the natural topology of sequences. This paper suggests that another topology may be more useful. It is obtained by taking $\mathrm{Lim}(W)$ as the derived set of W. The resulting topological space is shown to be completely regular, strongly zero-dimensional, and not normal. The languages appearing in the theory of ω-automata are shown to be closely related to functionally closed sets (continuous inverse images of the set $\{0\}$). As a consequence, a number of known results can be expressed and proved in terms of continuous mappings.

1. Introduction

Infinite words can be regarded as limits of finite words, and topology as an abstract study of limits. For this reason, languages with infinite words (ω-languages, ∞-languages) have been often studied as subsets of a topological space. Initially [6, 7, 11, 14, 17, 18, 23, 27, 28, 29, 30, 34, 36], this space was the set A^ω of infinite words over an alphabet A. Except in one case,[1] the set A^ω was always taken with the 'cartesian product' topology arising when A^ω is treated as the product of infinitely many copies of the discrete space A. This topology can also be introduced by defining the distance $d(x, y)$ between words x, y as a decreasing function of the length of their common prefix. It has open sets of the form VA^ω, where $V \subseteq A^*$,[2] and is sometimes called the 'natural topology' of A^ω [7, 28].

To study infinite words as limits of finite words, it is convenient to include both in the same space. This was done for the first time by Boasson and Nivat [4]. By

[1] In the earliest paper [14], infinite words were treated as expansions of real numbers.
[2] One may note that Choueka [8] considered a topology on the space A^*, where VA^* were the *closed* sets.

0304-3975/86/$3.50

treating the set $A^\infty = A^\omega \cup A^*$ as one topological space, they clearly exposed the topological character of notions such as adherence and center of a language.[3] The space A^∞ has been afterwards used in a number of papers [1, 2, 3, 5, 16, 21, 31]. In all cases, the topology of A^∞ was obtained by an obvious extension of the distance $d(x, y)$ to arbitrary words $x, y \in A^\infty$. The topology thus induced has open sets of the form $U \cup VA^\infty$, where $U, V \subseteq A^*$. In the following it will be called the *natural topology* of the space A^∞.

A closer examination shows that the notion of limit induced by the natural topology is different from that used in the theory of ω-automata. There are several concepts of limit associated with a topology. The most important one, according to [15], is that of a *limit point* (more often called an *accumulation point* or *cluster point*). In the natural topology, a word x is a limit point of a set W if it has infinitely many prefixes in common with members of W; thus, for example, a^ω is a limit point of the set $\{a^i b \mid i \geqslant 0\}$. In the theory of ω-automata, limits are introduced as upper bounds of infinite chains ordered by the relation of being a prefix (e.g., [11, p. 359]). The word a^ω is clearly not a limit of $\{a^i b \mid i \geqslant 0\}$ in this sense.

The present paper is motivated by the idea that a topology specifically chosen to reflect this second notion of limit may be more useful than the traditional one. One way to construct such a topology is to require that, for each set W, the limit points of W are identical to the upper bounds of all infinite chains contained in W. This requirement defines a unique topology, in the following denoted by $\boldsymbol{T}$.

Another way to imitate the intended notion of limit would be to start with the limit of a convergent sequence. An infinite chain could be regarded as an infinite sequence converging to its upper bound. We might require that all ascending sequences converge to their upper bounds and that no other sequences are convergent. Such a requirement, however, cannot be satisfied by any topology: any convergent sequence will remain convergent after its terms have been re-ordered. One could at best take the largest topology in which all ascending sequences converge (as the largest topology corresponds to the smallest class of convergent sequences). We show, in Section 7, that the topology defined in this way is the same as $\boldsymbol{T}$.

The purpose of this paper is to examine the properties of $\boldsymbol{T}$. It turns out that the space A^∞ with the topology $\boldsymbol{T}$ is completely regular, but not normal. We also show it to be strongly zero-dimensional and locally compact. The interesting fact seems to be that the sets usually studied in connection to ω-automata are closely related to functionally closed sets of $\boldsymbol{T}$ (continuous inverse images of the set $\{0\}$, zero-sets). As a consequence, a number of known results can be expressed and proved in terms of continuous mappings. Even if this does not uncover new facts, it certainly provides a new insight.

[3] Adherence and centre were introduced earlier by Nivat [19, 20] in a nontopological manner. A similar definition of centre, called Anf, was independently given by Prodinger and Urbanek [22], and extended to ω-languages by Freund [13].

2. Definitions and notation

The set of all natural numbers (nonnegative integers) is denoted by $\mathbb{N}$. For $k \in \mathbb{N} \cup \{\omega\}$, the set of first k natural numbers is denoted by $[k]$. Thus, in particular, $[0] = \emptyset$ and $[\omega] = \mathbb{N}$.

We consider an alphabet A, nonempty and at most countable. A function $x : [k] \to A$, where $k \in \mathbb{N} \cup \{\omega\}$, is called a *word* of *length k*. The length of a word x is denoted by $\lg(x)$. A word x is called *finite* if $\lg(x) \in \mathbb{N}$ and *infinite* if $\lg(x) = \omega$. The set of all words is denoted by A^∞, the set of all finite words by A^*, and the set of all infinite words by A^ω. (Traditionally, the elements of A^∞ have been also called ∞-*words*, and the elements of A^ω ω-*words*.) In the topological context, A^∞ is often referred to as a *space* and its elements as *points*. As remarked by Landweber [18], it is often useful to visualize A^∞ as an infinite tree, the words corresponding to finite and infinite paths from the root.

For $x, y \in A^\infty$, we write $x \leqslant y$, and say that x is a *prefix* of y, to mean that $\lg(x) \leqslant \lg(y)$ and $x(i) = y(i)$ for all $i \in [\lg(x)]$. It is easy to see that $\leqslant$ is a partial order in A^∞. As usual, we write $x < y$ to mean that $x \leqslant y$ and $x \neq y$. For $x \in A^\infty$ and $n \in \mathbb{N}$, we define

$$P_n(x) = \{y \in A^\infty \mid y \leqslant x \text{ and } \lg(y) \geqslant n\}.$$

A set $W \subseteq A^\infty$ is called a *chain* if, for each pair $x, y \in W$, either $x \leqslant y$ or $y \leqslant x$. It is easy to see that each infinite chain has a unique upper bound and each chain has the least upper bound. The least upper bound of a chain W is denoted by $\mathrm{Sup}(W)$.

A subset $W \subseteq A^\infty$ is also called a *language*. (Traditionally, subsets of A^∞ have been called ∞-*languages*, and subsets of A^ω ω-*languages*.) We apply the usual notation (concatenation, $*$, $^\omega$) to write down specific languages. As it is common, we often write x instead of $\{x\}$ for single-element sets. The complement of W with respect to A^∞ is denoted by CW. For $W \subseteq A^\infty$, we define

$$\mathrm{Fin}(W) = W \cap A^*, \qquad \mathrm{Inf}(W) = W \cap A^\omega.$$

Intervals on the line of real numbers are written in the usual way, with a square or round bracket indicating whether or not the end point is included in the interval; for example, $[0, a) = \{r \mid 0 \leqslant r < a\}$. The interval $[0, 1]$ with the natural topology of real numbers is denoted by I.

For a word $w \in A^\infty$ and a set $W \subseteq A^\infty$, we define the functions $\rho_w : A^\infty \to I$ and $\eta_W : A^\infty \to I$ as follows:

$$\rho_w(x) = \begin{cases} 0 & \text{for } x = w, \\ 2^{-\lg(x)-1} & \text{for } x < w, \\ 1 & \text{for } x \notin P_0(w); \end{cases}$$

$$\eta_W(x) = \begin{cases} 0 & \text{if } P_0(x) \cap W \text{ is infinite}, \\ 2^{-n} & \text{if } P_0(x) \cap W \text{ has } n \in \mathbb{N} \text{ elements}. \end{cases}$$

The following facts can be easily verified.

Property 2.1. *For each* $w \in A^\infty$, $0 < a \leqslant 1$, *and* $0 < b < 1$,

(1) $\rho_w^{-1}([0, a)) = P_{n-1}(w)$,

(2) $\rho_w^{-1}([0, b]) = P_{m-1}(w)$,

where $m, n \in \mathbb{N}$ *are the smallest numbers such that* $2^{-n} < a$ *and* $2^{-m} \leqslant b$.

Property 2.2. *For each* $W \subseteq A^*$, $0 < a \leqslant 1$, *and* $0 < b < 1$,

(1) $\eta_W^{-1}([0, a)) = \mathrm{Fin}(\eta_W^{-1}(2^{-n}))A^\infty$,

(2) $\eta_W^{-1}([0, b]) = \mathrm{Fin}(\eta_W^{-1}(2^{-m}))A^\infty$,

where $m, n \in \mathbb{N}$ *are the smallest numbers such that* $2^{-n} < a$ *and* $2^{-m} \leqslant b$.

The topological aspect of this paper is based on [10, 12].

3. Operators Lim and Clm

For $W \subseteq A^\infty$, we define

$$\mathrm{Lim}(W) = \{\mathrm{Sup}(U) \mid U \subseteq W \text{ is an infinite chain}\},$$

$$\mathrm{Clm}(W) = A^\omega - \mathrm{Lim}(CW).$$

These symbols can be read as the *limit* and the *colimit* of W, respectively. The set $\mathrm{Lim}(W)$ has been known in the literature under a number of different names: $\lim W$, $\mathrm{Lim}(W)$ in [9, 21, 32, 33], closure of W, $\bar{W}$ in [11, 20], G_2-set with G_2-base W in [18], δ-limit, W^δ in [28], $\mathfrak{G}_\delta(W)$ in [30, 35, 36], and W^e in [31]. It has been traditionally defined for $W \subseteq A^*$ only; since $\mathrm{Lim}(W)$ is fully defined by $\mathrm{Fin}(W)$, our extension to $W \subseteq A^\infty$ is just a notational convenience. A separate symbol for the set $\mathrm{Clm}(W)$ will be useful for expressing the dualities between open sets and closed sets.

The following facts can be easily verified.

Property 3.1. *For all* V, $W \subseteq A^\infty$,

(1) $\mathrm{Lim}(W) = \{x \in A^\omega \mid (P_0(x) \cap W)$ *is infinite*$\}$,

(2) $\mathrm{Clm}(W) = \{x \in A^\omega \mid (P_0(x) - W)$ *is finite*$\}$,

(3) $\mathrm{Lim}\,(W) = \mathrm{Lim}\,(\mathrm{Fin}\,(W))$,

(4) $\mathrm{Clm}\,(W) = \mathrm{Clm}(\mathrm{Fin}(W))$,

(5) $\mathrm{Lim}(\emptyset) = \mathrm{Clm}(\emptyset) = \mathrm{Clm}(\emptyset) = \emptyset$,

(6) $\mathrm{Lim}(A^*) = \mathrm{Clm}(A^*) = A^\omega$,

(7) $\mathrm{Clm}(W) \subseteq \mathrm{Lim}(W) \subseteq A^\omega$,

(8) $\mathrm{Lim}(V \cup W) = \mathrm{Lim}(V) \cup \mathrm{Lim}(W)$,

(9) $\mathrm{Clm}(V \cap W) = \mathrm{Clm}(V) \cap \mathrm{Clm}(W)$,

(10) $\mathrm{Lim}(W) = \eta_W^{-1}(0)$.

Under the assumption that A is at most countable, the set A^* is countable. As soon as A contains more than one letter, the set A^ω is not countable, and, because of Property 3.1(3), there exist subsets of A^ω that cannot be represented as $\mathrm{Lim}(W)$ for any W. We define

$$\mathbf{Lim} = \{\mathrm{Lim}(W) \mid W \subseteq A^*\},$$

$$\mathbf{Clm} = \{\mathrm{Clm}(W) \mid W \subseteq A^*\} = \{A^\omega - \mathrm{Lim}(W) \mid W \subseteq A^*\}.$$

For $A = \{a, b\}$, an example of a set not belonging to **Lim** is $W = A^*a^\omega$ [18, Lemma 3.1]; on the other hand, $A^\omega - W = (a^*b)^\omega = \mathrm{Lim}(A^*b) \in \mathbf{Lim}$. By definition, we have thus $W \in \mathbf{Clm}$ and $A^\omega - W \notin \mathbf{Clm}$. It is easy to see that $aA^\omega \in \mathbf{Lim} \cap \mathbf{Clm}$. Thus, all three classes: $\mathbf{Lim} - \mathbf{Clm}$, $\mathbf{Clm} - \mathbf{Lim}$, and $\mathbf{Lim} \cap \mathbf{Clm}$ are in general not empty.

4. The topology $\boldsymbol{T}$: open sets and closed sets

We note the following properties of Lim (where $V, W \subseteq A^\infty$ and $x \in A^\infty$):
(1) $\mathrm{Lim}(\emptyset) = \emptyset$,
(2) $\mathrm{Lim}(\mathrm{Lim}(W)) \subseteq W \cup \mathrm{Lim}(W)$,
(3) $\mathrm{Lim}(V \cup W) = \mathrm{Lim}(V) \cup \mathrm{Lim}(W)$,
(4) $x \notin \mathrm{Lim}(x)$.
These properties guarantee that there exists a unique topology in which the set of limit points of W (the *derived set* of W) is equal to $\mathrm{Lim}(W)$ [10, p. 73]. We denote this topology by $\boldsymbol{T}$, and call it the *Lim-topology* of A^∞. The topology $\boldsymbol{T}$ is implied throughout the rest of the discussion, unless otherwise stated. As usual, we identify $\boldsymbol{T}$ with the family of its open sets.

We start by characterizing the closed sets and open sets of $\boldsymbol{T}$.

Property 4.1. *The following conditions are equivalent for each* $W \subseteq A^\infty$:
(1) *W is a closed set,*
(2) $\mathrm{Lim}(W) \subseteq \mathrm{Inf}(W)$,
(3) *for each* $x \in \mathrm{Inf}(CW)$, *there exists* $n \in \mathbb{N}$ *such that* $P_n(x) \subseteq CW$.

Proof. A set is closed if and only if it contains its derived set; since $\mathrm{Lim}(W) \subseteq A^\omega$ for each W, this is equivalent to (2). By Property 3.1(1), (2) is equivalent to saying that each infinite word outside W has only finitely many prefixes in W, which is expressed by (3). $\square$

Property 4.2. *The following conditions are equivalent for each* $W \subseteq A^\infty$:
(1) *W is an open set,*
(2) $\mathrm{Inf}(W) \subseteq \mathrm{Clm}(W)$,
(3) *For each* $x \in \mathrm{Inf}(W)$, *there exists* $n \in \mathbb{N}$ *such that* $P_n(x) \subseteq W$.

Proof. A set is open if and only if it is a complement of a closed set. Using the definition of Clm, one can verify that (2) is equivalent to $\mathrm{Lim}(\mathbf{C}W) \subseteq \mathrm{Inf}(\mathbf{C}W)$ which, according to Property 4.1(2), states that $\mathbf{C}W$ is closed. By referring to Property 4.1(3), one may verify that (3) states the same thing. □

A set is called *open-and-closed* if it is both open and closed.

Property 4.3. *W is open-and-closed iff* $\mathrm{Inf}(W) = \mathrm{Lim}(W) = \mathrm{Clm}(W)$.

Proof. The proof is straightforward from Properties 4.1(2), 4.2(2), and $\mathrm{Clm}(W) \subseteq \mathrm{Lim}(W)$. □

It follows that, in particular, each $W \subseteq A^\omega$, each $W \supseteq A^\omega$, and each finite $W \subseteq A^\infty$ are closed, while each $W \subseteq A^*$ and each $W \supseteq A^*$ are open. Each $P_n(x)$, for $x \in A^\infty$ and $n \in \mathbb{N}$, and each set of the form VA^∞, with $V \subseteq A^*$, are open-and-closed. We note that each set of the form $U \cup VA^\infty$, with $U, V \subseteq A^*$, is open; the topology $\boldsymbol{T}$ is thus larger than the natural topology.

For $W \subseteq A^\infty$, its *closure*, denoted $\mathrm{Cl}(W)$, is the union of W and its derived set. The *interior* of W is the set $\mathrm{In}(W) = \mathbf{C}\mathrm{Cl}(\mathbf{C}W)$, and the *boundary* (or *frontier*) of W is the set $\mathrm{Fr}(W) = \mathrm{Cl}(W) - \mathrm{In}(W)$. The following facts are easy to verify using Property 3.1.

Property 4.4. *For each* $W \subseteq A^\infty$,
(1) $\mathrm{Cl}(W) = \mathrm{Fin}(W) \cup (\mathrm{Inf}(W) \cup \mathrm{Lim}(W))$,
(2) $\mathrm{In}(W) = \mathrm{Fin}(W) \cup (\mathrm{Inf}(W) \cap \mathrm{Clm}(W))$,
(3) $\mathrm{Fr}(W) = (\mathrm{Inf}(W) - \mathrm{Lim}(W)) \cup (\mathrm{Lim}(W) - \mathrm{Clm}(W)) \cup (\mathrm{Clm}(W) - \mathrm{Inf}(W))$,
(4) $\mathrm{Cl}(\mathrm{In}(W)) = \mathrm{Fin}(W) \cup \mathrm{Lim}(W)$,
(5) $\mathrm{In}(\mathrm{Cl}(W)) = \mathrm{Fin}(W) \cup \mathrm{Clm}(W)$.

A set $W \subseteq A^\infty$ is a *closed domain* if $W = \mathrm{Cl}(\mathrm{In}(W))$, and an *open domain* if $W = \mathrm{In}(\mathrm{Cl}(W))$ [12, p. 37]. From Property 4.4(4) and (5) we have the following property.

Property 4.5. *For each* $W \subseteq A^\infty$,
(1) *W is a closed domain if and only if* $\mathrm{Inf}(W) = \mathrm{Lim}(W)$,
(2) *W is an open domain if and only if* $\mathrm{Inf}(W) = \mathrm{Clm}(W)$.

5. Basis

For $x \in A^\infty$, a *basis at the point* x is a family of open sets containing x, such that each open set $G \ni x$ contains a member of the family.

Property 5.1. *For each $x \in A^{\infty}$, the family $\boldsymbol{B}_x = \{P_n(x) \mid n \in \mathbb{N}, n \leqslant \lg(x)\}$ is a basis at x.*

Proof. As remarked before, each set of the form $P_n(x)$ is open. If $x \in A^*$, the set $P_{\lg(x)}(x) = \{x\} \in \boldsymbol{B}_x$ is contained in each $G \subseteq x$. If $x \in A^{\omega}$, $\boldsymbol{B}_x$ has the required property by Property 4.2(3). □

From Property 2.1, the following property is obtained.

Property 5.2. *For each $x \in A^{\infty}$, the family $\{\rho_x^{-1}([0, a)) \mid 0 < a \leqslant 1\}$ is identical to $\boldsymbol{B}_x$, and thus constitutes a basis at x.*

A *basis* for $\boldsymbol{T}$ is a family of open sets such that each set $G \in \boldsymbol{T}$ is a union of members of the family. Such a basis can, in particular, be obtained as the union of bases at all points. We thus have the following property.

Property 5.3. *The family $\boldsymbol{B} = \bigcup\{\boldsymbol{B}_x \mid x \in A^{\infty}\}$ is a basis for $\boldsymbol{T}$.*

A *subbasis* for $\boldsymbol{T}$ is a family of open sets such that each set $G \in \boldsymbol{T}$ can be represented in terms of arbitrary unions and finite intersections of members of the family.

Property 5.4. *The family $\boldsymbol{B}' = \bigcup\{\boldsymbol{B}_x \mid x \in A^{\omega}\}$ is a subbasis for $\boldsymbol{T}$ (unless A consists of a single letter).*

Proof. It is enough to show that each set of the form $\{x\}$, where $x \in A^*$, is a finite intersection of members of $\boldsymbol{B}'$. Indeed, each such set is an intersection of $P_n(xa^{\omega})$ and $P_n(xb^{\omega})$, where $a, b \in A$ and $n = \lg(x)$. □

We note that the space A^{∞} is *first-countable*, that is, has a countable basis at each point. We also note that $\boldsymbol{T}$ has a basis consisting of open-and-closed sets.

6. Continuous mappings

Let Z be any topological space. A function $f: A^{\infty} \to Z$ is *continuous* if $f^{-1}(Y)$ is open for each open set $Y \subseteq Z$. The following alternative definitions of continuity are well known [10, p. 79].

Property 6.1. *Each of the following conditions is equivalent to a function $f: A^{\infty} \to Z$ being continuous*:

(1) *$f^{-1}(Y)$ is open for each member Y of a subbasis for Z,*

(2) *$f^{-1}(Y)$ is closed for each closed $Y \subseteq Z$,*

(3) *f is continuous at each point $x \in A^{\infty}$, that is, for each member Y of the basis at $f(x)$, there exists a member B of the basis at x such that $f(B) \subseteq Y$.*

In the following, continuous functions from A^∞ to I will be of a special interest. We have, in particular, the following property.

Property 6.2. *All functions* $\rho_w : A^\infty \to I$ *for* $w \in A^\infty$ *and all functions* $\eta_W : A^\infty \to I$ *for* $W \subseteq A^*$ *are continuous.*

Proof. According to Property 6.1(1), it is enough to show that the inverse images of the intervals $[0, a)$ and $(b, 1]$ are open for $0 < a \leqslant 1$ and $0 \leqslant b < 1$ (since these intervals constitute a subbasis for I). According to Properties 2.1(1) and 2.2(1), the inverse images of $[0, a)$ are open for all the stated values at a. According to Properties 2.1(2) and 2.2(2), the complements of inverse images of $(b, 1]$ are closed for $0 < b < 1$. For $b = 0$, we have $\rho_w^{-1}((b, 1]) = \boldsymbol{C}\{w\}$ and $\eta_W^{-1}((b, 1]) = \boldsymbol{C}\,\mathrm{Lim}(W)$, which are both open. □

Since $\{x\}$ is open for each $x \in A^*$, $f : A^\infty \to Z$ is always continuous at each point $x \in A^*$. From Properties 5.1 and 6.1(3), we obtain the following property.

Propperty 6.3. *A function* $f : A^\infty \to I$ *is continuous if and only if for each* $x \in A^\omega$ *and* $\varepsilon > 0$ *there exists* $n \in \mathbb{N}$ *such that* $|f(x) - f(y)| < \varepsilon$ *whenever* $y \in P_n(x)$.

It is well known [10, p. 84] that if $f : A^\infty \to I$ and $g : A^\infty \to I$ are continuous, so are their absolute values, sum, difference, product, and quotient (this latter under the condition that the denominator is never 0).

Another interesting class of continuous functions are functions from A^∞ to A^∞. From Properties 5.1 and 6.1(3), we obtain the following property.

Property 6.4. *A function* $f : A^\infty \to A^\infty$ *is continuous if and only if, for each* $x \in A^\omega$ *and natural number* $m \leqslant \lg(f(x))$, *there exists an* $n \in \mathbb{N}$ *such that* $f(y) \in P_m(f(x))$ *whenever* $y \in P_n(x)$.

Using Property 5.2, one can imitate the classical ε-δ-form.

Property 6.5. *A function* $f : A^\infty \to A^\infty$ *is continuous if and only if for each* $x \in A^\omega$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $\rho_{f(x)}(f(y)) < \varepsilon$ *whenever* $\rho_x(y) < \delta$.

A special case of functions from A^∞ to A^∞ are the *monotone* functions. A function $f : A^\infty \to A^\infty$ is monotone if, for every $x, y \in A^\infty$, $x \leqslant y$ implies $f(x) \leqslant f(y)$.

Property 6.6. *The following conditions are equivalent for a monotone function* $f : A^\infty \to A^\infty$:

(1) *f is continuous,*

(2) $f^{-1}(z)$ *is closed for each* $z \in A^*$,

(3) $\mathrm{Sup}(f(W)) = f(\mathrm{Sup}(W))$ *for each infinite chain* W.

Proof. (1)⇒(2): This implication follows from Property 6.1(2).

(2)⇒(3): Let $W \subseteq A^\infty$ be an infinite chain. If f is monotone, $f(W)$ is also a chain and $f(\mathrm{Sup}(W))$ is its upper bound. From this follows $\mathrm{Sup}(f(W)) \leqslant f(\mathrm{Sup}(W))$. If $\mathrm{Sup}(f(W))$ is infinite, the only possibility is $\mathrm{Sup}(f(W)) = f(\mathrm{Sup}(W))$. If $\mathrm{Sup}(f(W))$ is finite, $f(W)$ must be a finite set; if (2) holds, $f^{-1}(f(W)) \supseteq W$ is closed as a finite union of closed sets. Being closed, it must contain $\mathrm{Sup}(W)$; thus, $f(\mathrm{Sup}(W)) \in f(W)$, which gives $f(\mathrm{Sup}(W)) \leqslant \mathrm{Sup}(f(W))$.

(3)⇒(1): Suppose f is not continuous. Then, by Property 6.4, there exists $x \in A^\omega$ such that $f(y) \notin P_m(f(x))$ for infinitely many words $y \leqslant x$. All these words y clearly form an infinite chain W with $\mathrm{Sup}(W) = x$. If f is monotone, $f(W)$ is also a chain and is contained in $P_0(f(x))$. By definition, $f(W)$ is disjoint with $P_m(f(x))$, and thus must be a finite set. This implies $\mathrm{Sup}(f(W)) \in f(W)$; Since $f(\mathrm{Sup}(W)) = f(x) \in P_m(f(x))$, $\mathrm{Sup}(f(W)) \neq f(\mathrm{Sup}(W))$. □

We note that Property 6.6(3) is formally identical to the definition of continuity used by Scott [25, 26]. However, as A^∞ is not a lattice, Sup is defined only for chains, and Property 6.6(3) makes sense only for the functions f that transform chains into chains. What we have shown is that, in the domain of such functions, our definition of continuity indeed coincides with that of Scott. A similar property of the natural topology was observed in [4].

7. Sequences and convergence

Let Z be any topological space. A function $\varphi : \mathbb{N} \to Z$ is called a *sequence* of elements of Z. The sequence φ *converges to* $x \in Z$, written $\varphi \mapsto x$, if, for every open set $G \supseteq x$, there exists an $n \in \mathbb{N}$ such that $\varphi(i) \in G$ for all $i \geqslant n$. The point x is then called the *limit* of φ.

Property 7.1. *A sequence $\varphi : \mathbb{N} \to A^\infty$ converges to $x \in A^\infty$ if and only if, for each $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ such that $\rho_x(\varphi(i)) < \varepsilon$ for all $i \geqslant n$.*

Proof. It is easy to see that one obtains an equivalent definition of convergence if the set $G \supseteq x$ is replaced by a member of the basis at x. The property then follows from Property 5.2. □

Other characterizations of convergence have been given in [24]; they seem to be of little interest here. As a digression, we note that the set $\mathbb{N}' = \mathbb{N} \cup \{\omega\}$ could be regarded as the set of all words over a one-letter alphabet. With Lim-topology assumed for $\mathbb{N}'$, Property 7.1 becomes similar to Property 6.5, and states that $\varphi : \mathbb{N} \to A^\infty$ converges if and only if φ has a continuous extension to $\mathbb{N}'$; the limit of φ is then the value of the extension for ω.

It follows from Property 7.1 that if a sequence φ of words converges to x, all elements of φ, from some point on, must be prefixes of x. The length of $\varphi(i)$ must approach that of x as i increases; however, the elements of φ need not be ordered according to length.

Let a sequence φ of words be called *ascending* if $\varphi(i) < \varphi(i+1)$ for each $i \in \mathbb{N}$. It is easy to see that, for each ascending sequence φ, the set $\varphi(\mathbb{N})$ of all its terms is a chain. From Property 7.1 the next property immediately follows.

Property 7.2. *Each ascending sequence* $\varphi : \mathbb{N} \to A^\infty$ *converges to* $\mathrm{Sup}(\varphi(\mathbb{N}))$.

As mentioned in Section 1, we have the following property.

Property 7.3. *$\boldsymbol{T}$ is the largest topology on A^∞ having Property* 7.2.

Proof. Let $\boldsymbol{T}'$ be any topology on A^∞. We show that $\boldsymbol{T}'$ satisfying Property 7.2 implies $\boldsymbol{T}' \subseteq \boldsymbol{T}$.

Suppose $\boldsymbol{T}' \not\subseteq \boldsymbol{T}$, that is, there exists a set G that is open in $\boldsymbol{T}'$, but not in $\boldsymbol{T}$. By Property 5.3, G is not the union of members of $\boldsymbol{B}$. That means, there exists $x \in G$ such that $x \in B \subseteq G$ is not satisfied by any $B \in \boldsymbol{B}$. Clearly, $x \in A^\omega$, for, otherwise, we would have $x \in \{x\} \subseteq G$. Let φ be the ascending sequence of all prefixes of x. Consider any $n \in \mathbb{N}$. The set $P_n(x)$, being a member of $\boldsymbol{B}$ and containing x, must contain an element not belonging to G. That means, there exists $i \geqslant n$ such that $\varphi(i) \notin G$. This shows that, in $\boldsymbol{T}'$, φ does not converge to x, and $\boldsymbol{T}'$ does not satisfy Property 7.2. □

Because A^∞ is first-countable, all of its elementary concepts can be characterized in terms of convergent sequences. We thus have the following properties [10, p. 218].

Property 7.4. *A set $W \in A^\infty$ is closed if and only if it contains the limit of each convergent sequence $\varphi : \mathbb{N} \to W$.*

Property 7.5. *Let Z be any topological space. A function $f : A^\infty \to Z$ is continuous at $z \in A^\infty$ if and only if for each sequence φ of words, $\varphi \mapsto x$ implies $f \circ \varphi \mapsto f(x)$.*

A useful variation of the above is formulated in the following property.

Property 7.6. *A function $f : A^\infty \to Z$ is continuous at $z \in A^\omega$ if and only if for the ascending sequence φ of all prefixes of x, $f \circ \varphi$ converges to $f(x)$.*

Proof. (1) The condition is necessary by Properties 7.2 and 7.5.

(2) To show that it is sufficient, consider any open set Y, $f(x) \in Y \subseteq Z$. If $f \circ \varphi$ converges to $f(x)$, there exists $n \in \mathbb{N}$ such that $f \circ \varphi(i) \in Y$ for all $i \geqslant n$. However, that means $f(P_n(x)) \subseteq Y$, and f is continuous at x by Property 6.1(3). □

8. Functionally open and functionally closed sets

A set $W \subseteq A^\infty$ is *functionally closed* (or a *zero-set*) if there exists a continuous function $f: A^\infty \to I$ such that $W = f^{-1}(0)$ [12, p. 64]. The complement of a functionally closed set is *functionally open* (or a *cozero set*). Each functionally closed set is closed, and each functionally open set is open. Each open-and-closed set is both functionally open and functionally closed. The finite unions and countable intersections of functionally closed sets are functionally closed; the finite intersections and countable unions of functionally open sets are functionally open.

Property 8.1. *A set $W \subseteq A^\omega$ is a member of* **Lim** *if and only if it is functionally closed.*

Proof. (1) The condition is necessary by Properties 3.1(10) and 6.2.

(2) The condition is sufficient: Consider any $W \subseteq A^\omega$ such that $W = f^{-1}(0)$ for a continuous $f: A^\infty \to I$. Denote $V_n = \min(f^{-1}((2^{-n-1}, 2^{-n}]))$, where $\min(U)$ is defined as the set of all such $x \in U$ that $y \notin U$ for all $y < x$. We show that $W = \text{Lim}(V)$, where $V = \bigcup \{V_n \mid n \in \mathbb{N}\}$.

(a) $W \subseteq \text{Lim}(V)$. Suppose $x \in W$, and choose any $k \in \mathbb{N}$. By Property 6.3, there exists an m such that $y \in P_m(x) \Rightarrow |f(y) - f(x)| < 2^{-k} \Leftrightarrow f(y) < 2^{-k}$. Choose any $y \in P_m(x) - \{x\}$. Since $W \subseteq A^\omega$, y is not in W, and thus, $f(y) \neq 0$; there must exist an integer $n > k$ such that $2^{-n-1} < f(y) \leqslant 2^{-n}$. Hence, x has at least one prefix in $f^{-1}((2^{-n-1}, 2^{-n}])$; the shortest such prefix is in V_n. This shows that, for any $k \in \mathbb{N}$, x has a prefix in V_n for some $n > k$, and thus in V_n for infinitely many values of n. Since all sets V_n are disjoint, this means $x \in \text{Lim}(V)$.

(b) $W \supseteq \text{Lim}(V)$. Suppose $x \notin W$, that is, $f(x) \neq 0$. *Let* $\varepsilon = \frac{1}{2} f(x)$. Choose any $k \in \mathbb{N}$ such that $2^{-k} < \varepsilon$. By Property 6.3, there exists an m such that $y \in P_m(x) \Rightarrow |f(x) - f(y)| < \varepsilon \Leftrightarrow f(y) \geqslant \varepsilon > 2^{-k}$. Thus, at most m prefixes of x can be in $\bigcup \{V_n \mid n \geqslant k\}$. As each V_n can contain at most one prefix of x, only finitely many of them can be in $\bigcup \{V_n \mid n < k\}$. Hence, x has only finitely many prefixes in V, and $x \notin \text{Lim}(V)$. $\square$

Property 8.2. *A set $W \subseteq A^\infty$ is functionally closed if and only if it is a closed set with* $\text{Inf}(W) \in$ **Lim.**

Proof. (1) The condition is necessary: Let W be functionally closed; as stated before, each such set is closed. Since $A^\omega = \text{Lim}(A^*)$, A^ω is functionally closed by Property 8.1. The set $\text{Inf}(W)$ is thus functionally closed as an intersection of two functionally closed sets. From Property 8.1 follows $\text{Inf}(W) \in$ **Lim.**

(2) The condition is sufficient: Suppose W is closed and $\text{Inf}(W) = \text{Lim}(V)$ for some $V \subseteq A^*$. Define

$$f(x) = \begin{cases} 0 & \text{for } x \in \text{Fin}(W), \\ \eta_V(x) & \text{for } x \notin \text{Fin}(W). \end{cases}$$

It is easy to see that $f^{-1}(0) = W$. It remains to show that f is continuous; for this purpose, we use Property 6.3. Consider any $x \in A^\omega$ and $\varepsilon > 0$. By continuity of η_V, there exists $n \in \mathbb{N}$ such that $y \in P_n(x) \Rightarrow |\eta_V(x) - \eta_V(y)| < \varepsilon$.

Case 1: $x \in \mathrm{Lim}(W)$. Since W is closed, we have $\mathrm{Lim}(W) \subseteq \mathrm{Lim}(V)$, which means $x \in \mathrm{Lim}(V)$ and $\eta_V(x) = f(x) = 0$. Clearly, $f(y) \leqslant \eta_V(y)$ for each $y \in A^\infty$; thus, $y \in P_n(x) \Rightarrow |0 - f(y)| < |0 - \eta_V(y)| < \varepsilon$.

Case 2: $x \notin \mathrm{Lim}(W)$. Then, at most finitely many prefixes of x are in $\mathrm{Fin}(W)$, and there exists $k \in \mathbb{N}$ such that $f(y) = \eta_V$ for all $y \in P_k(x)$. Taking $m = \max(k, n)$, we have $y \in P_m(x) \Rightarrow |f(x) - f(y)| = |\eta_V(x) - \eta_V(y)| < \varepsilon$. $\square$

From part (2) of the above proof the next property immediately follows.

Property 8.3. *Each functionally closed set W can be represented as $W = f^{-1}(0)$ for a continuous function $f: A^\infty \to I$ with values in the set* $\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$.

By taking the dual forms of the above results, we obtain the following properties.

Property 8.4. *A set $W \subseteq A^\omega$ is a member of* **Clm** *if and only if $W \cup A^*$ is functionally open.*

Property 8.5. *A set $W \subseteq A^\infty$ is functionally open if and only if it is an open set with* $\mathrm{Inf}(W) \in$ **Clm**.

Property 8.6. *Each functionally open set W can be represented as $W = f^{-1}((0, 1])$ for a continuous function $f: A^\infty \to I$ with values in the set* $\{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\}$.

The characterization of classes **Lim** and **Clm** in terms of functionally closed and functionally open sets (Properties 8.1 and 8.4) is considered by the author to be the most interesting result of this paper. The ω-languages accepted by finite-state automata according to different schemes are intersections and unions of members of **Lim** and **Clm**. The characterization of these classes in terms of G_δ-sets and F_σ-sets of the natural topology seems to account for most of the topological results concerning ω-automata [18, 29, 30, 31, 36]. Functionally open and functionally closed sets have useful properties that will enable us, in Section 10, to prove a number of known results in terms of these sets, and thus, indirectly, in terms of continuous functions.

In fact, the functionally closed sets of $\boldsymbol{T}$ are very similar to G_δ-sets of the natural topology. A set W is a G_δ-set of the natural topology if and only if $\mathrm{Inf}(W) \in$ **Lim**; a functionally closed set of $\boldsymbol{T}$ has to satisfy the additional condition that $\mathrm{Lim}(W) \subseteq \mathrm{Inf}(W)$. A corresponding relationship exists between the functionally open sets of $\boldsymbol{T}$ and F_σ-sets of the natural topology.

For the sake of completeness, we note two other characterizations of the class **Lim** in terms of $\boldsymbol{T}$: as the class of all derived sets, and as the class of infinite-word parts of closed domains. The first one is trivial: the topology T itself was so defined; the other is very closely related to this definition. Both classes lack the useful properties of functionally closed sets.

9. Separation and disconnectedness

Two sets W_1, $W_2 \subseteq A^\infty$ have *disjoint neighbourhoods* if there exists disjoint open sets $G_1 \supseteq W_1$ and $G_2 \supseteq W_2$.

Property 9.1. *The following conditions are equivalent for a pair of sets W_1, $W_2 \subseteq A^\infty$:*

(1) *W_1 and W_2 have disjoint neighborhoods;*

(2) *There exist disjoint functionally open sets $U_1 \supseteq W_1$ and $U_2 \supseteq W_2$;*

(3) *There exists a set $U \subseteq A^*$ such that $W_1 \subseteq U \cup \mathrm{Clm}(U) \subseteq U \cup \mathrm{Lim}(U) \subseteq CW_2$.*

Proof. (1)$\Rightarrow$(2): Suppose $G_1 \supseteq W_1$ and $G_2 \supseteq W_2$ are open and disjoint. Using $G_1 \subseteq CG_2$ and Property 3.1, we obtain $\mathrm{Clm}(G_1) \subseteq \mathrm{Lim}(G_1) \subseteq A^\omega - \mathrm{Clm}(G_2)$. From this and Property 4.2(2) follows:

$$
\begin{aligned}
G_1 &\subseteq \mathrm{Fin}(G_1) \cup \mathrm{Clm}(G_1) \subseteq \mathrm{Fin}(G_1) \cup \mathrm{Lim}(G_1) \\
&\subseteq C(\mathrm{Fin}(G_2) \cup \mathrm{Clm}(G_2)) \subseteq CG_2.
\end{aligned}
$$

By Properties 4.2(2) and 8.5, the sets $U_1 = \mathrm{Fin}(G_1) \cup \mathrm{Clm}(G_1)$ and $U_2 = \mathrm{Fin}(G_1) \cup \mathrm{Clm}(G_2)$ are functionally open; they clearly satisfy (2).

(2)$\Rightarrow$(3): Since all functionally open sets are open, the above applies also when G_1, G_2 are already functionally open. The set $U = \mathrm{Fin}(G_1)$ clearly satisfies (3).

(3)$\Rightarrow$(1): Let U be as stated by (3). Using Properties 3.1 and 4.2(2), one can easily verify that the sets $G_1 = U \cup \mathrm{Clm}(U)$ and $G_2 = C(U \cup \mathrm{Lim}(U))$ are open and disjoint; by (3), they contain, respectively, W_1 and W_2. $\square$

Two sets W_1, $W_2 \subseteq A^\infty$ are *completely separated* if there exists a continuous function $f: A^\infty \to I$ such that $f(x) = 0$ for all $x \in W_1$ and $f(x) = 1$ for all $x \in W_2$. The function f is said to *completely separate* W_1 and W_2 [12, p. 64].

Property 9.2. *The following conditions are equivalent for a pair of sets W_1, $W_2 \subseteq A^\infty$:*

(1) *W_1 and W_2 are completely separated;*

(2) *There exist disjoint functionally closed sets $F_1 \supseteq W_1$ and $F_2 \supseteq W_2$;*

(3) *There exists an open-and-closed set U such that $W_1 \subseteq U \subseteq CW_2$;*

(4) *There exists a set $U \subseteq A^*$ such that $W_1 \subseteq U \cup \mathrm{Clm}(U) = U \cup \mathrm{Lim}(U) \subseteq CW_2$.*

Proof. (1)$\Rightarrow$(2): Let W_1 and W_2 be completely separated by the function f. Define $g(x) = 1 - f(x)$; the function $g: A^\infty \to I$ is clearly continuous. It is easy to see that the sets $F_1 = f^{-1}(0)$ and $F_2 = g^{-1}(0)$ satisfy (2).

(2)⇒(3): Let $F_1 \supseteq W_1$ and $F_2 \supseteq W_2$ be as stated by (2). By Property 8.3, $F_1 = f^{-1}(0)$ and $F_2 = g^{-1}(0)$ for some continuous, rational-valued functions $f: A^\infty \to I$ and $g: A^\infty \to I$. Define

$$h(x) = f(x)/(f(x)+g(x)).$$

Since F_1 and F_2 are disjoint, $f(x)+g(x) \neq 0$ for all $x \in A^\infty$, and $h: A^\infty \to I$ is everywhere defined and continuous. Since both f and g are rational-valued, so is h. Let $a \in I$ be an irrational number, for example, $a = 2^{-1/2}$. Define

$$U_0 = h^{-1}(a), \qquad U_1 = h^{-1}([0, a)), \qquad U_2 = h^{-1}((a, 1]).$$

Clearly, $U_0 \cup U_1 \cup U_2 = A^\infty$. Since h is rational-valued, we have $U_0 = \emptyset$, and the sets U_1, U_2 form a partition of A^∞. Being inverse images of open sets, they are both open; being complements of each other, they are both closed. It is easy to see that $W_1 \subseteq F_1 = h^{-1}(0) \subseteq U_1$ and $W_2 \subseteq F_2 = h^{-1}(1) \subseteq U_2$. Denoting $U = U_1$, we have $W_1 \subseteq U \subseteq CW_2$.

(3)⇒(1): Let U be as stated by (3). Let $f(x) = 1$ for $x \in U$ and $f(x) = 0$ for $x \notin U$. For an open-and-closed set U, the function $f: A^\infty \to I$ so defined is continuous. It completely separates W_1 and W_2.

(3)⇔(4): This follows from Property 4.3. □

It follows from Properties 9.1(3) and 9.2(4) that two completely separated sets always have disjoint neighbourhoods. The converse is not necessarily true; an example are the sets $W_1 = A^*a \cup A^*a^\omega$ and $W_2 = A^*b \cup A^*b^\omega$ (where $A = \{a, b\}$).

It follows further from Properties 9.2(2) and (3) that, for each set $W \in \mathbf{Lim} \cap \mathbf{Clm}$, there exists an open-and-closed set U such that $\mathrm{Inf}(U) = W$. Together with Property 4.3, this gives the following topological characterization of the class $\mathbf{Lim} \cap \mathbf{Clm}$.

Property 9.3. *A set $W \subseteq A^\omega$ is a member of* $\mathbf{Lim} \cap \mathbf{Clm}$ *if and only if $W = \mathrm{Inf}(U)$ for an open-and-closed set U.*

After these preliminary results, we proceed to classify the space A^∞ according to its separation and disconnectedness properties.

A topological space is a T_1-*space* if for every pair of distinct points there exists an open set containing one of them but not the other. This property is equivalent to all single-point sets being closed. Since this is the case for A^∞, we have the following property.

Property 9.4. *A^∞ is a T_1-space.*

A T_1-space is *completely regular* (or *Tychonoff*) if, for each closed set $F \subseteq A^\infty$, and point $x \notin F$, $\{x\}$ and F are completely separated. According to Propertty 9.2(3), this is in particular true if the space has a basis consisting of open-and-closed sets. We thus have the following property.

Property 9.5. *A^∞ is a completely regular (Tychonoff) space.*

Being completely regular, A^∞ is also a *separated* (or *Hausdorff*) space (that is, all distinct points $x, y \in A^\infty$ have disjoint neighbourhoods).

A T_1-space is *normal* if each pair of disjoint closed sets have disjoint neighbourhoods.

Property 9.6. *A^∞ is not a normal space.*

Proof. Consider any set $W_1 \subseteq A^\omega$; let $W_2 = A^\omega - W_1$. W_1 and W_2 are clearly closed and disjoint. Suppose they have disjoint neighbourhoods. Then, by Property 9.1(3), there exists U such that $\mathrm{Inf}(W_1) \subseteq \mathrm{Clm}(U) \subseteq \mathrm{Lim}(U) \subseteq A^\omega - \mathrm{Inf}(W_2)$. By $\mathrm{Inf}(W_1) = W_1 = A^\omega - \mathrm{Inf}(W_2)$, this means $W_1 = \mathrm{Clm}(U) = \mathrm{Lim}(U)$. Hence, W_1 and W_2 do not have disjoint neighbourhoods unless they are both members of $\mathbf{Lim} \cap \mathbf{Clm}$. $\square$

A Tychonoff space is *strongly zero-dimensional* if for each pair of completely separated sets W_1 and W_2 there exists an open-and-closed set U such that $W_1 \subseteq U \subseteq CW_2$ [12, p. 444]. Thus, according to Property 9.2(3), we have the following property.

Property 9.7. *A^∞ is strongly zero-dimensional.*

10. Some infinite unions and intersections

We start with countable unions and intersections.

Property 10.1. *The following conditions are equivalent for each $W \subseteq A^\infty$:*

(1) *W is functionally closed;*

(2) *W is a countable intersection of functionally closed sets;*

(3) *W is a countable intersection of open-and-closed sets;*

(4) *W is a countable intersection of closed sets of the form $U \cup VA^\infty$, where $U, V \subseteq A^*$;*

(5) *W is an intersection of a sequence of sets $F_0 \supseteq G_0 \supseteq F_1 \supseteq G_1 \supseteq \cdots$, where F_n is closed and G_n is open for each $n \in \mathbb{N}$.*

Proof. The proof follows the pattern (1)⇒(4)⇒(3)⇒(2)⇒(1) and (1)⇒(5)⇒(3).

(1)⇒(4): Let W be functionally closed. By Properties 8.2 and 3.1(10), W is closed and equal to $\mathrm{Fin}(W) \cup \eta_V^{-1}(0)$, where $V \subseteq A^*$ is such that $\mathrm{Lim}(V) = \mathrm{Inf}(W)$. For $n \in \mathbb{N}$, define

$$F_n = \mathrm{Fin}(W) \cup \eta_V^{-1}([0, 2^{-n}]).$$

Since W is closed, we have $\mathrm{Lim}(W) \subseteq \mathrm{Lim}(V) \subseteq \eta_V^{-1}([0, 2^{-n}])$ and F_n is also closed. According to Property 2.2, for each $n > 0$ we have $F_n = \mathrm{Fin}(W) \cup V_n A^\infty$, where $V_n \subseteq A^*$. It is easy to see that $W = \bigcap \{F_n \mid n > 0\}$.

(4)⇒(3)⇒(2): Closed sets of the form $U \cup VA^\infty$ are open-and-closed; open-and-closed sets are functionally closed.

(2)$\Rightarrow$(1): Countable intersections of functionally closed sets are functionally closed.

(1)$\Rightarrow$(5): Let $W=f^{-1}(0)$ for a continuous $f: A^{\infty} \to I$. For $n \in \mathbb{N}$, define

$$F_n = f^{-1}([0, 2^{-n}]), \qquad G_n = f^{-1}([0, 2^{-n})).$$

It is easy to see that these sets satisfy (5).

(5)$\Rightarrow$(3): Let W be an intersection of sets F_n, G_n as stated. By Property 9.1(2), for each $n \in \mathbb{N}$ there exist a functionally closed set F'_n and a functionally open set G'_n such that $F_n \supseteq F'_n \supseteq G'_n \supseteq G_n$. It is easy to see that W is an intersection of the sequence $F'_0 \supseteq G'_0 \supseteq F'_1 \supseteq G'_1 \supseteq \cdots$.

By Property 9.2(2) and (3), for each $n \in \mathbb{N}$ there exists an open-and-closed set H_n such that $G'_n \supseteq H_n \supseteq F'_{n+1}$. It is easy to see that $W = \bigcap \{H_n \mid n \in \mathbb{N}\}$. □

A property dual to the above is the following.

Property 10.2. *The following conditions are equivalent for each $W \subseteq A^{\infty}$:*

(1) *W is functionally open;*

(2) *W is a countable union of functionally open sets;*

(3) *W is a countable union of open-and-closed sets;*

(4) *W is a countable union of open sets of the form $C(U \cup VA^{\infty})$, where $U, V \subseteq A^*$;*

(5) *W is a union of a sequence of sets $G_0 \subseteq F_0 \subseteq G_1 \subseteq F_1 \subseteq \cdots$, where G_n is open and F_n is closed for each $n \in \mathbb{N}$.*

Let $\boldsymbol{G}_\delta$ denote the class of all countable intersections of functionally open sets and $\boldsymbol{F}_\sigma$ the class of all countable unions of functionally closed sets. Using Properties 10.1(3) and 10.2(3), one can verify, in a rather standard way, the following property.

Property 10.3. *The class $\boldsymbol{G}_\delta \cap \boldsymbol{F}_\sigma$ is a Boolean algebra containing all functionally open and all functionally closed sets.*

By intersecting the sets appearing in Properties 10.1 and 10.2 with A^{ω}, and applying Properties 8.1, 8.4, and 9.3, we obtain the following properties.

Property 10.4. *The following conditions are equivalent for each $W \subseteq A^{\omega}$:*

(1) *$W \in$ **Lim**;*

(2) *W is a countable intersection of members of* **Lim***;*

(3) *W is a countable intersection of members of* **Lim** $\cap$ **Clm***;*

(4) *W is a countable intersection of sets of the form VA^{ω}, where $V \subseteq A^*$.*

Property 10.5. *The following conditions are equivalent for each $W \subseteq A^{\omega}$:*

(1) *$W \in$ **Clm**;*

(2) *W is a countable union of members of* **Clm***;*

(3) *W is a countable union of members of* **Lim** $\cap$ **Clm***;*

(4) *W is a countable union of sets of the form $C(VA^{\omega})$, where $V \subseteq A^*$.*

All these are classical results, obtained in a new way in terms of $\boldsymbol{T}$. This new way, however, is not entirely independent of the traditional one. The key to the traditional results is [18, Lemma 2.2], that links **Lim** to G_δ-sets of the natural topology. The main lines of its proof, disguised in the terminology of continuous functions, can still be recognized in the proofs of Property 8.1 (part (2)) and of Property 10.1 ((1)⇒(4)). Another classical result [18, Theorem 3.3] can be obtained by restricting Property 10.3 to the subspace A^ω.

Properties 10.1(5) and 10.2(5) do not seem to have a known counterpart.

We now proceed to arbitrary unions and intersections.

Property 10.6. *The following conditions are equivalent for each* $W \subseteq A^\infty$:

(1) *W is closed*;

(2) *W is an intersection of functionally closed sets*;

(3) *W is an intersection of open-and-closed sets*;

(4) *W is an intersection of closed sets of the form $U \cup VA^\infty$ where $U, V \subseteq A^*$.*

Proof. (1)⇒(3): $\boldsymbol{T}$ has a basis consisting of open-and-closed sets.

(3)⇒(2): Each open-and-closed set is functionally closed.

(2)⇒(4): This follows from Property 10.1(4).

(4)⇒(1): Intersection of closed sets is closed. □

A property dual to the above is the following.

Property 10.7. *The following conditions are equivalent for each* $W \subseteq A^\infty$:

(1) *W is open*;

(2) *W is a union of functionally open sets*;

(3) *W is a union of open-and-closed sets*;

(4) *W is a union of open sets of the form $\boldsymbol{C}(U \cup VA^\infty)$, where $U, V \subseteq A^*$.*

Finally, we have the following property.

Property 10.8. *The following holds for each set* $W \subseteq A^\infty$:

(1) *W is a G_δ-set, that is, a countable intersection of open sets*;

(2) *W is a F_σ-set, that is, a countable union of closed sets*;

(3) *W is a union of functionally closed sets*;

(4) *W is an intersection of functionally open sets.*

Proof. (1) Let be $W \subseteq A^\omega$. For each $n \in \mathbb{N}$, define $G_n = \bigcup \{P_n(x) \mid x \in W\}$. It can be easily verified that $W = \bigcap \{G_n \mid n \in \mathbb{N}\}$. Each G_n is clearly an open set; hence, each $W \subseteq A^\omega$ is a G_δ-set. Each open set is a G_δ-set, and thus, in particular, each $W \subseteq A^*$. As finite unions of G_δ-sets are G_δ-sets, so is each $W = \mathrm{Fin}(W) \cup \mathrm{Inf}(W) \subseteq A^\infty$.

(2) This follows from F_σ-sets being complements of G_δ-sets.

(3) For each $x \in A^\infty$, the set $\{x\}$ is closed, and $\mathrm{Inf}(\{x\}) \in \mathbf{Lim}$; thus, $\{x\}$ is functionally closed by Property 8.2. Each set W is clearly a union of single-point sets.

(4) is dual to (3). □

We note that, according to Property 10.8(1), (2), the Borel sets of T are totally uninteresting.

11. Retracts

A continuous function $r: A^\infty \to A^\infty$ having the property that $r \circ r = r$ is called a *retraction*; the image $r(A^\infty)$ is then called a *retract* (of A^∞). Retracts have a number of useful properties, for example, that every continuous function defined on a retract is extendable to A^∞ [10, p. 323].

Property 11.1. *A nonempty set $R \subseteq A^\infty$ is a retract if and only if it is closed and the set* $\mathrm{Inf}(R) - \mathrm{Lim}(R)$ *is a countable member of* $\mathbf{Lim} \cap \mathbf{Clm}$.

Proof. (1) The condition is necessary: Let R be a retract and r the corresponding retraction. Each retract of a Hausdorff space is closed [10, p. 322]; thus, R is closed and $\mathrm{Lim}(R) \subseteq \mathrm{Inf}(R)$.

Denote $W = \mathrm{Inf}(R) - \mathrm{Lim}(R)$ and consider any $w \in W$. Since $w \notin \mathrm{Lim}(R)$, there exists an n such that $P_n(w) \cap R = \{w\}$. It is easy to see that $r^{-1}(V) = r^{-1}(V \cap R)$ for each $V \subseteq A^\infty$. Thus, $r^{-1}(w) = r^{-1}(P_n(w) \cap R) = r^{-1}(P_n(w))$, and $r^{-1}(w)$ is open-and-closed (being equal to the inverse image of an open-and-closed set). The sets $r^{-1}(w)$ for $w \in W$ are all mutually disjoint and nonempty; each of them, being open, contains finite words. Since A^* is countable, there can be at most countably many such sets; in other words, W must be countable. Being countable, W can be represented as $W = \{w_0, w_1, w_2, \ldots\}$ (the proof for finite W is analogous). For $i \in \mathbb{N}$, denote $U_i = r^{-1}(w_i)$. Using Properties 9.5 and 9.7, and the fact that U_i is open, one can partition U_i into two open sets, X_i and Y_i, such that $\mathrm{Inf}(X_i) = w_i$.

The unions $X = \bigcup\{X_i \mid i \in \mathbb{N}\}$ and $Y = \bigcup\{Y_i \mid i \in \mathbb{N}\}$ are also open and constitute a partition of $r^{-1}(W) = \bigcup\{U_i \mid i \in \mathbb{N}\}$. The set $r^{-1}(W)$ is open as a union of open sets and closed as an inverse image of a closed set. From this follows that both X and Y are open-and-closed. It is easy to see that $W = \mathrm{Inf}(X)$; $W \in \mathbf{Lim} \cap \mathbf{Clm}$ follows from Property 4.3.

(2) The condition is sufficient: Let R be as stated. Denote $W = \mathrm{Inf}(R) - \mathrm{Lim}(R)$. Let $W = \{w_0, w_1, w_2, \ldots\}$. From $W \in \mathbf{Lim} \cap \mathbf{Clm}$, it follows that the sets W and $(A^\omega - W) \cup \mathrm{Fin}(R)$ are both functionally closed. By Property 9.2(2) and (3), there exists an open-and-closed set U such that $W \subseteq U \subseteq C((A^\omega - W) \cup \mathrm{Fin}(R))$. Define

$$G_0 = P_0(w_0) \cap U,$$

$$G_i = P_0(w_i) \cap (U - \bigcup\{G_n \mid n < i\}) \quad \text{for } i > 0.$$

One can verify by induction that G_i is open-and-closed and $G_i \cap R = w_i$ for each $i \in \mathbb{N}$. It is easy to see that all sets G_i are disjoint.

Let z be any fixed element of R. Define the function $r: A^\infty \to A^\infty$ as follows:

(a) if $x \in R$, then $r(x) = x$;

(b) if $x \notin R$ and $x \in G_i$ for some $i \in \mathbb{N}$, then $r(x) = w_i$;

(c) if $x \notin R$ and $x \notin G_i$ for any $i \in \mathbb{N}$ and x has a prefix in R, then $r(x)$ is the longest such prefix;

(d) otherwise, $r(x) = z$.

It is easy to see that r maps A^∞ onto R and is an identity within R. It remains to show that r is continuous. For this purpose, we use Property 6.4. Consider any $x \in A^\omega$ and choose any $m \leq \lg(r(x))$. The following cases are possible:

Case 1: $x \in W$, that is, $x = w_i$ for some i. Since G_i is open, there exists an n such that $P_n(x) \subseteq G_i$. Consider any $y \in P_n(x)$. By (a) and (b), we have: $r(y) = w_i \in P_m(w_i) = P_m(r(x))$. In all the remaining cases, we have $x \notin W$, and thus $x \in CU$. Since CU is open, there exists a k such that $P_k(x) \subseteq CU$, and thus $P_k(x) \cap G_i = \emptyset$ for all $i \in \mathbb{N}$. In the following, k is assumed to have this property.

Case 2: $x \in \mathrm{Lim}(R)$. As x has infinitely many prefixes in R, R must contain $u < x$ such that $\lg(u) > \max(m, k)$. Let $n = \lg(u)$. Consider any $y \in P_n(x)$. If $y \in R$, we have $r(y) = y \in P_m(x) = P_m(r(x))$ by (a) and $n > m$. If $y \notin R$, y has a prefix v in R, $u \leq v < x$. By (a), $n > k$, and (c), we have $r(y) = v \in P_m(x) = P_m(r(x))$.

Case 3: $x \notin R$ and x has a prefix in R. Since $x \notin \mathrm{Lim}(R)$, x has the longest prefix u belonging to R. Let $n = \max(\lg(u), k)$. Consider any $y \in P_n(x)$. By $n > k$ and (c), we have $r(y) = u \in P_m(u) = P_m(r(x))$.

Case 4: $x \notin R$ and x has no prefix in R. Choose any $n \in \mathbb{N}$ and $y \in P_n(x)$. By (d). we have $r(y) = z \in P_m(z) = P_m(r(x))$. $\square$

The above result provides one more characterization of the class Lim, as the class of infinite-word parts of retracts.

12. Pseudometrics

Since each metric space is normal, the topology $\boldsymbol{T}$ cannot be induced by any metric. However, since A^∞ is completely regular, $\boldsymbol{T}$ can be induced by a family of pseudometrics [10, p. 198] (other terms for pseudometric are *gauge* or *écart*). One such family can be constructed as follows.

Choose a word $w \in A^\infty$. For this word w and any $x, y \in A^\infty$, define

$$d_w(x, y) = |\rho_w(x) - \rho_w(y)|.$$

It is easy to see that d_w has all the properties of a pseudometric, namely,

$$d_w(x, x) = 0,$$

$$d_w(x, y) = d_w(y, x),$$

$$d_w(x, y) + d_w(y, z) \geq d_w(x, z),$$

for all $x, y, z \in A^\infty$. A d_w-*ball* with centre $x \in A^\infty$ and radius $r > 0$ is defined as

$$B(x; d_w, r) = \{y \in A^\infty \mid d_w(x, y) < r\}.$$

Define $\boldsymbol{D} = \{d_w \mid w \in A^\infty\}$. The family $\boldsymbol{D}$ of pseudometrics induces the topology $\boldsymbol{T}$ in the following sense.

Property 12.1. *The family* $\boldsymbol{B}(\boldsymbol{D}) = \{B(x; d_w, r) \mid x \in A^\infty, d_w \in \boldsymbol{D}, r > 0\}$ *constitutes a basis for* $\boldsymbol{T}$.

Proof. It is enough to show that (1) all members of $\boldsymbol{B}(\boldsymbol{D})$ are open, and (2) $\boldsymbol{B} \subseteq \boldsymbol{B}(\boldsymbol{D})$.

(1) For fixed $w, x \in A^\infty$, define $f(y) = d_w(x, y)$. Being an absolute value of the difference between the constant $\rho_w(x)$ and the continuous function $\rho_w(y)$, the function $f: A^\infty \to I$ is continuous. The ball $B(x; d_w, r)$ can be defined in terms of f as $f^{-1}([0, r))$; it is open being an inverse image of an open set under a continuous mapping.

(2) According to Property 5.2, each element of $\boldsymbol{B}$ can be expressed as $\rho_w^{-1}([0, r))$ for $0 < r \leqslant 1$. Since $\rho_w(w) = 0$, $\rho_w^{-1}([0, r))$ is a d_w-ball with centre w and radius r. $\square$

Another family of pseudometrics is obtained by defining

$$d_W(x, y) = |\eta_W(x) - \eta_W(y)|,$$

for a fixed set $W \subseteq A^*$. The family $\{d_W \mid W \subseteq A^*\}$ also induces $\boldsymbol{T}$; the proof is similar to that for $\boldsymbol{D}$, except that construction of $\boldsymbol{B}$ as the set of open d_W-balls is less straightforward. (The set W used to express $P_n(x)$ for $x \in A^\infty$ is $zA^\infty - P_{n+1}(x)$, where z is the shortest word in $P_n(x)$.)

13. Compactness

A Hausdorff space is *compact* if each covering of the space by open sets has a finite subcovering. It is *locally compact* if, for each point x, there exists an open set $G \supseteq x$ such that each open covering of $\mathrm{Cl}(G)$ has a finite subcovering.

Since each compact space is normal, A^∞ is not compact. However, we have the following property.

Property 13.1. A^∞ *is locally compact.*

Proof. For $x \in A^*$, we have $G = \mathrm{Cl}(G) = \{x\}$: $\{x\}$ has exactly one, finite, covering.

For $x \in A^\omega$, we have $G = \mathrm{Cl}(G) = P_0(x)$: by Property 4.2(3), the member of the covering that contains x must contain $P_n(x)$ for some n; this leaves out at most n elements, that can be covered by at most n members of the covering. $\square$

References

[1] A. Arnold, Sémantique des processus communicants, *RAIRO Inform. Théor.* **15** (1981) 103–139.

[2] A. Arnold, Topological characterizations of infinite behaviours of transition systems, in: J. Diaz, ed., *Automata, Languages and Programming*, Lecture Notes in Computer Science **154** (Springer, Berlin, 1983) 28–38.

[3] J. Beauquier and M. Nivat, Application of formal language theory to problems of security and synchronization, in: R. V. Book, ed., *Formal Language Theory: Perspectives and Open Problems* (Academic Press, New York/London, 1980) 407–454.

[4] L. Boasson, and M. Nivat, Adherences of languages, *J. Comput. System Sci.* **20** (1980) 285-309.

[5] L. Boasson and M. Nivat, Centers of languages, in: P. Deussen, ed., *Theoretical Computer Science*, Lecture Notes in Computer Science **104** (Springer, Berlin, 1981) 245-251.

[6] J.R. Büchi and L.H. Landweber, Definability in the monadic second order theory of successor, *J. Symbolic Logic* **34** (1969) 166-170.

[7] J.R. Büchi and L.H. Landweber, Solving sequential conditions by finite-state strategies, *Trans. Amer. Math. Soc.* **138** (1969) 295-311.

[8] Y. Choueka, Structure automata, *IEEE Trans. Comput.* **C-23** (1974) 1218-1227.

[9] Y. Choueka, Theories of automata on ω-tapes: A simplified approach, *J. Comput. System. Sci.* **8** (1974) 117-141.

[10] J. Dugundji, *Topology* (Allyn and Bacon, Rockleigh, 1966).

[11] S. Eilenberg, *Automata, Languages and Machines Vol. A* (Academic Press, New York/London, 1974).

[12] R. Engelking, *General Topology* (Polish Scientific Publishers (PWN), Warsaw, 1977).

[13] R. Freund, Init and Anf operating on ω-languages, *Inform. Process. Lett.* **16** (1983) 265-269.

[14] J. Hartmanis and R.E. Stearns, Sets of numbers defined by finite automata, *Amer. Math. Monthly* **74** (1967) 539-542.

[15] J.G. Hocking and G.S. Young, *Topology* (Addison-Wesley, Reading, MA, 1961).

[16] S. Istrail, Some remarks on non-algebraic adherences, *Theoret. Comput. Sci.* **21** (1982) 341-349.

[17] K. Kobayashi, M. Takahashi and H. Yamasaki, Characterization of ω-regular languages by first-order formulas, *Theoret. Comput. Sci.* **28** (1984) 315-327.

[18] L.H. Landweber, Decision problems for ω-automata, *Math. Systems Theory* **3** (1969) 376-384.

[19] M. Nivat, Mots infinis engendrés par une grammaire algébrique, *RAIRO Inform. Théor.* **11** (1977) 311-327.

[20] M. Nivat, Sur les ensembles des mots infinis engendrés par une grammaire algébrique, *RAIRO Inform. Théor.* **12** (1978) 259-278.

[21] D. Niwinski, Fixed-point characterization of context-free ∞-languages, *Inform. and Control* **61** (1984) 247-277.

[22] H. Prodinger and F.J. Urbanek, Language operators related to Init, *Theoret. Comput. Sci.* **8** (1979) 161-175.

[23] M.O. Rabin, Decidability of second-order theories and automata on infinite trees, *Trans. Amer. Math. Soc.* **141** (1969) 1-35.

[24] R.R. Redziejowski, An alternative topology for the study of ω-languages, Tech. Rept. TR 18.230, IBM Nordic Laboratory, Lidingö, Sweden, 1984.

[25] D. Scott, The lattice of flow-diagrams, in: E. Engeler, ed., *Symposium on Semantics of Algorithmic Languages*, Lecture Notes in Mathematics **188** (Springer, Berlin, 1971) 311-366.

[26] D. Scott, Data types as lattices, *SIAM J. Comput.* **5** (1976) 522-587.

[27] L. Staiger, Reguläre Nullmengen, *Elektron. Informationsverarb. Kybernet.* **12** (1976) 307-311.

[28] L. Staiger, A note on connected ω-languages, *Elektron. Informationsverarb. Kybernet.* **16** (1980) 245-251.

[29] L. Staiger, Finite-state ω-languages, *J. Comput. System Sci.* **27** (1983) 434-448.

[30] L. Staiger and K. Wagner, Automatentheoretische und automatenfreie Charakterisierungen topologischer Klassen regulärer Folgenmengen, *Elektron. Informationsverarb. Kybernet.* **10** (1974) 379-392.

[31] M. Takahashi and H. Yamasaki, A note on ω-regular languages, *Theoret. Comput. Sci.* **23** (1983) 217-225.

[32] W. Thomas, Star-free regular sets of ω-sequences, *Inform. and Control* **42** (1979) 148-156.

[33] W. Thomas, A combinatorial approach to the theory of ω-automata, *Inform. and Control* **48** (1981) 261-283.

[34] B.A. Trakhtenbrot and Y.M. Barzdin, *Finite Automata, Behavior and Synthesis* (North-Holland, Amsterdam, 1973).

[35] K. Wagner, Eine Axiomatisierung der Theorie de regulären Folgenmengen, *Elektron. Informationsverarb. Kybernet.* **12** (1976) 337-354.

[36] K. Wagner, On ω-regular sets, *Inform. and Control* **43** (1979) 123-177.